147. On Torus Cohomotopy Groups

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1. The main object of this note is an application of my theorem in the note [1]. Torus homotopy groups are defined by Fox [2], [3]; but in this note I have adopted another meaning of the torus, and the methods of the paper are strongly influenced by Spanier's paper [4].

2. In this section and the followings, I will use the definitions and lemmas of my note [1], which we refer to as [D].

Lemma 2.1. Let (X, A) be a compact pair with dim $(X-A) < 4n-1$. If $\alpha, \beta, \alpha', \beta' : (X, A) \to (T^{2n}, q)$ with $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ and if $g : (X, A) \to (T^{2n} \vee T^{2n}, (q, q))$ is a normalization of $\alpha \times \beta$ and g': $(X, A) \to (T^{2n} \vee T^{2n}, (q, q))$ is a normalization of $\alpha' \times \beta'$, then $\Omega g \simeq \Omega g'$.

Proof. Since $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$, $\alpha \times \beta \simeq \alpha' \times \beta'$. Then $g \simeq \alpha \times \beta \simeq \alpha' \times \beta' \simeq g'$. Hence, there is a map

$$F : (X \times I, A \times I) \to (T^{2n} \times T^{2n}, (q, q))$$

such that

$$\begin{aligned} F(x, 0) &= g(x) \\ F(x, 1) &= g'(x) \end{aligned} \quad \text{for all } x \in X.$$

Then $(X \times 0) \cup (X \times 1) \subset F^{-1}(T^{2n} \vee T^{2n})$, by [$D$], Lemma 2.3, dim $M < 4n$ for any closed $M \subset X \times I - A \times I$. Hence by [D] Theorem 3.5, a normalization G of F exists such that $G(x, t) = F(x, t)$ for $(x, t) \in F^{-1}(T^{2n} \vee T^{2n})$. That is, there is a map

$$G : (X \times I, A \times I) \to (T^{2n} \vee T^{2n}, (q, q))$$

such that

$$\begin{aligned} G(x, 0) &= F(x, 0) = g(x) \\ G(x, 1) &= F(x, 1) = g'(x) \end{aligned} \quad \text{for all } x \in X.$$

Then $\Omega G : (X \times I, A \times I) \to (T^{2n}, q)$ is a homotopy between Ωg and $\Omega g'$.

Theorem 2.2. If (X, A) is a compact pair with dim $(X-A) < 2n-1$, the homotopy classes $\{\alpha\}$ of maps α of (X, A) into (T^{2n}, q) form an abelian group with the law of composition $\{\alpha\} + \{\beta\} = \{\alpha < f > \beta\}$, where f is an arbitrary normalization of $\alpha \times \beta$.

Proof. [D] Theorem 3.5 implies that a normalization f of $\alpha \times \beta$ exists. Lemma 2.1 of the present note shows that $\{\alpha < f > \beta\}$ does not depend on the choice of $\alpha \in \{\alpha\}$, $\beta \in \{\beta\}$ nor upon the normalization f involved. Therefore, the class $\{\alpha < f > \beta\}$ is uniquely determined by the class $\{\alpha\}$ and $\{\beta\}$.