36. Vector-space Valued Functions on Semi-groups. II

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In an earlier Note (5),*) the author developed theory of vector-valued functions, especially almost periodic functions and ergodic functions on a semi-group G into a locally convex vector space E and proved the existence theorem of the mean value of ergodic function for some spaces.

In this Note, we shall consider a locally convex vector space E such that every ergodic function $f(x)$ has the mean $M(f)$. Therefore, there is an $M(f)$ of E such that, for any n.b.d. U,

$$M(f)-\frac{1}{n}\sum_{i=1}^{n}f(a_i d)\in U$$

and

$$M(f)-\frac{1}{m}\sum_{j=1}^{m}f(cb_j)\in U$$

for some $a_i (i=1,2,\dots,n)$, $b_j (j=1,2,\dots,m)$ and all c, d of G.

III. Invariant linear space of ergodic functions

The following propositions are clear.

Proposition 3.1. A constant function $f(x)\equiv f$ has the mean f: $M(f)=f$.

Proposition 3.2. If $f(x)$ is ergodic, then $\alpha f(x)$ is ergodic and

$$M(\alpha f)=\alpha M(f).$$

Definition 3. Let $\mathfrak{M}$ be a set of ergodic functions. $\mathfrak{M}$ is said to be a left invariant linear set, if it satisfies the following conditions:
(3) for any element a of G and $f(x)\in\mathfrak{M}$, $f(ax)\in\mathfrak{M}$,
(4) for any reals, α, β, and $f(x)$, $g(x)\in\mathfrak{M}$, $\alpha f(x)+\beta g(x)\in\mathfrak{M}$.

Theorem 8. Let $\mathfrak{M}$ be a left invariant linear set of ergodic function, then

(5) $$M_x(f(ax))=M_x(f(x)),$$

(6) $$M(\alpha f+\beta g)=\alpha M(f)+\beta M(g).$$

Proof. Let $f\in\mathfrak{M}$ and U any n.b.d., then there are elements $a_1, a_2,\dots,a_n$ and d of G such that

$$M_x(f(ax))-\frac{1}{n}\sum f(aa_i d)\in U.$$

Thus $M_x(f(ax))$ is U-left mean of $f(x)$. This proves (5).

We shall prove $M(f+g)=M(f)+M(g)$. Since the Propositions 1, 2, we have the equality (6). For a given n.b.d. U, we can find

*) Additional references are given in (5).