31. On Blocks of Characters of the Symmetric Group

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The following basic theorem in the modular representation theory of the symmetric group S_n has been proved in various ways (2, 5).

Theorem. *Two irreducible representations of S_n belong to the same block if and only if they have the same p-core.*

In the present paper we shall give a new proof of this theorem.

1. Let $[\alpha]$ be a Young diagram of n nodes which contains α_i nodes in its ith row and α'_j nodes in its jth column:

(1) $$n=\sum_i \alpha_i=\sum_j \alpha'_j.$$

We denote by χ_α the character of the irreducible representation of S_n associated with $[\alpha]$ and by f_α its degree.

The node in the ith row and jth column of $[\alpha]$ is called its ij-node. It is called the corner of the ij-right hook that consists of this node and all nodes to the right of it or below it. Let us denote by $h_{i,j}$ the total hook length of the ij-right hook. The hook product H_α of $[\alpha]$ is the product of the n integers $h_{i,j}$ (3). Then we have

(2) $$f_\alpha=n!/H_\alpha.$$

Lemma 1. *If the kl-right hook of length $h_{k,l}=g$ is removed from $[\alpha]$ leaving $[\gamma]$, then*

$$f_\alpha/f_\gamma=\frac{n!}{(n-g)!\,g!}KLM \text{ with } K=\prod_{i<k}((h_{i,l}-g)/h_{i,l}),$$

$$L=\prod_{j<l}((h_{k,j}-g)/h_{k,j}), \qquad M=\prod_{k<i\leqq\alpha'_l}((g-h_{i,l})/h_{i,l}).$$

Proof. We denote by $h'_{i,j}$ the total hook length of the ij-right hook of $[\gamma]$. We see easily that

(3) $$h'_{i,j}=\begin{cases} h_{i+1,j} & \text{if } k\leqq i<\alpha'_l,\ j<l, \\ h_{k,j}-g & \text{if } i=\alpha'_l,\ j<l, \\ h_{i,j+1} & \text{if } i<k,\ l\leqq j<\alpha_k, \\ h_{i,l}-g & \text{if } i<k,\ j=\alpha_k, \\ h_{i+1,j+1} & \text{if } k\leqq i,\ l\leqq j, \\ h_{i,j} & \text{otherwise.} \end{cases}$$

Moreover we have (3, Lemma 1)

(4) $$\prod_{l\leqq j\leqq\alpha_k} h_{k,j} \prod_{k<i\leqq\alpha'_l}(g-h_{i,l})=g!.$$

The lemma is proved easily by (2)–(4).

If we set $\beta_i=h_{i,1}$, $\beta'_j=h_{1,j}$, then we see that Lemma 1 is identical with the lemma (4, p. 101) since