25. *On the Convergence of Some Gap Series*

By Noboru MATSUYAMA and Shigeru TAKAHASHI
Mathematical Institute, Kanazawa University, Japan
(Comm. by Z. SUETUNA, M.J.A., March 12, 1955)

§ 1. Let $f(x)$, $-\infty<x<+\infty$, be a function satisfying the following conditions:

(1.1) $$f(x+1)=f(x),$$

and

(1.2) $$\int_0^1 f(x)dx=0, \qquad \int_0^1 f^2(x)dx=1.$$

Further, let us put

(1.3) $$\omega(n)=\left(\int_0^1 \left| f(x)-s_n(x)\right|^2 dx\right)^{1/2}$$

where $s_n(x)$ denotes the n-th partial sum of the Fourier series of $f(x)$.

The following theorems were proved for the sequence $\{n_k\}$ of integers which has the Hadamard gap.

Theorem of M. Kac, R. Salem, and A. Zygmund [1]. *If*

(1.4) $$\omega(n)=O(1/(\log n)^{\alpha}), \qquad \alpha>1 \qquad (n\to+\infty)$$

and

(1.5) $$\sum c_n^2(\log n)^2<\infty,$$

then the series

(1.6) $$\sum c_k f(n_k x)$$

converges almost everywhere.

Theorem of S. Izumi [2]. *If*

(1.7) $$\omega(n)=O(1/n^{\alpha}), \qquad \alpha>0 \qquad (n\to+\infty)$$

and

(1.8) $$\sum c_n^2(\log_2 n)^2<+\infty,$$

then (1.6) converges almost everywhere.

The purpose of this paper is to generalize above results. Following G. Alexits [3], we shall say that a sequence $\{a_n\}$ is $\lambda(n)$-lacunary if

(1.9) [the number of n's such that $a_n\neq 0$ for $2^k\leqq n<2^{k+1}$]$=O(\lambda(k))$ $(k\to+\infty)$,

where $\{\lambda(n)\}(n=0,1,2,\dots)$ is a non-decreasing sequence of positive numbers.

In the following, we shall assume that the sequence $\{a_n\}$ is $\lambda(n)$-lacunary and treat the convergence problem of the series

(1.10) $$\sum_{k=1}^{\infty} a_k f(kx).$$