49. Integrability of Trigonometrical Series. II

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1. We shall consider the trigonometrical series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1}$$

Given a sequence $c_0, c_1, c_{-1}, \dots$ such that $c_n \to 0$, let $c_0^* \geqq c_1^* \geqq c_{-1}^* \geqq c_2^* \geqq \cdots$ be the sequence $|c_0|, |c_1|, |c_{-1}|, \dots$ arranged in the descending order of magnitude.

Recently R. P. Boas [1] proved the following

Theorem B. *If $1 < q \leqq 2$, $1 \leqq p < q/(q-1)$, and $\alpha < 1 - q/p'$, then (1) is the Fourier series of a function of L^p if $c_n \to 0$ and*

$$\sum_{n=-\infty}^{\infty} |c_{n+m} - c_{n-m}|^q = O(m^\alpha) \tag{2}$$

as $m \to \infty$ through the multiples of some fixed integer.

If $\alpha \geqq 1 - q/p'$ the conclusion no longer holds.

In this paper we prove the following theorems.

Theorem 1. *If $q \geqq 2$, $p \geqq 1$, and $0 < \alpha < q/p - 1$, then (1) is the Fourier series of a function of L^p if $c_n \to 0$ and*

$$\sum_{n=-\infty}^{\infty} (c_{n+m} - c_{n-m})^{*q} n^{q-2} = O(m^\alpha) \tag{3}$$

as $m \to \infty$ through the multiples of some fixed integer.

If $\alpha = q/p - 1$, $\alpha > q - 2$, the conclusion no longer holds.

Theorem 2. *If $q \geqq 2$, $p \geqq 1$, $q' \leqq r \leqq q$, $\mu = 1/r + 1/q - 1$, and $0 < \alpha < q/p - 1$, then (1) is the Fourier series of a function of L^p if $c_n \to 0$ and*

$$\sum_{n=-\infty}^{\infty} |c_{n+m} - c_{n-m}|^r (|n|+1)^{-\mu r} = O(m^{\alpha r/q}) \tag{4}$$

as $m \to \infty$ through the multiples of some fixed integer.

If $\alpha \geqq q/p - 1$ the conclusion no longer holds.

In Theorem 2, if $r = q'$ then it becomes Theorem B, and if $r = q$ then it becomes Theorem 1 except star. Hence Theorem 2 contains Theorem B formally but Theorems 1 and 2 are mutually exclusive.

The proofs of Theorems 1 and 2 are similar to that of Theorem B, the difference being to use the following Theorems HL 1 and HL 2 [2], respectively, instead of the Hausdorff-Young theorem. We prove here Theorem 1 only.

Theorem HL 1. *If $q \geqq 2$ then (1) is the Fourier series of a function $f(x)$ of L^q and*