94. *Some Trigonometrical Series. XV*

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1. G. H. Hardy and J. E. Littlewood [1] have proved the following

Theorem 1. *Let* $p\geqq 1$, $0<\alpha<1$, *and* $\alpha p>1$. *If* $f(x)$ *belongs to the Lip* (α, p) *class, then* $f(x)$ *is equivalent to a function belonging to the Lip* $(\alpha-1/p)$ *class.*

This was generalized in the following form [2]:

Theorem 2. *Let* $p\geqq 1$, $0<\alpha<1$, *and* $\alpha p>1$. *If* $f(x)$ *belongs to the Lip* (α, p) *class, then*

(1) $$f(x)-s_n(x)=O(1/n^{\alpha-1/p})$$

uniformly almost everywhere, where $s_n(x)$ *is the nth partial sum of the Fourier series of* $f(x)$.

If is well known that (1) implies that $f(x)$ is equivalent to a function of the Lip $(\alpha-1/p)$ class.

In the proof of Theorem 2 in [2], Theorem 1 is used. We shall prove here Theorem 2, without using Theorem 1, but using the idea in [1]. From this proof we get the following

Theorem 3. *Let* $p\geqq 1$, $0<\alpha<1$, *and* $\alpha p<1$. *If* $f(x)$ *belongs to the Lip* (α, p) *class, then*

(2) $$s_n(x)-f(x)=O(n^{1/p-\alpha})$$

uniformly almost everywhere.

It is known that if $f(x)$ belongs to the L^p class ($p\geqq 1$), then

$$s_n(x)=o(n^{1/p})$$

and the exponent $1/p$ is the best possible one. Estimation of the integral mean of the left side of (2) was given by E. S. Quade [3].

2. We shall prove Theorem 2. Let $f(x)$ be a function in the Lip (α, p) class, and let

$$\varphi_x(t)=f(x+t)+f(x-1)-2f(x)$$

and $s_n(x)$ be the nth partial sum of the Fourier series of $f(t)$ at $t=x$. Then (cf. [4])

$$|s_n(x)-f(x)|\leqq\int_{\pi/n}^{\pi}\frac{\varphi_x(t)-\varphi_x(t+\pi/n)}{t^2}dt$$
$$+\frac{A}{n}\int_{\pi/n}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt+2n\int_0^{2\pi/n}|\varphi_x(t)|\,dt+O(1/n^{\alpha}).$$

If we prove that