165. On Coverings and Continuous Functions of Topological Spaces

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The purpose of this paper is to study relations between continuous functions and locally finite coverings playing the important rôle in recent topological developments. We shall establish a necessary and sufficient condition for a normal space to be fully normal and a condition for metrizability by using families of continuous functions and shall generalize Hausdorff's extension theorem of continuous function by using coverings.

Lemma. *Let R be a topological space and $V_\alpha=\{x \mid f_\alpha(x)>0\}$*[1] *$(\alpha<\tau)$, where $f_\alpha(\alpha-\tau)$ are real valued functions on R. If $\mathfrak{V}=\{V_\alpha \mid \alpha<\tau\}$ is a covering of R, and if $\underset{\beta<\alpha}{\smile} f_\alpha(x)$ is continuous for every $\alpha<\tau$, then $\mathfrak{V}$ has a locally finite refinement.*

Proof. Let $V_{1\alpha}=\left\{x \mid f_\alpha(x)>\frac{1}{2}\right\}$ and $V_{i\alpha}=\left\{x \mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}\right\}$ $(n\geqq 2)$, then $\overline{V}_{i\alpha}\subseteq V_{i+1\alpha}\subseteq V_\alpha$ $(i=1,2\ldots)$.

Define $N_{n1}=V_{n1}$, $N_{n\alpha}=V_{n\alpha}-\overline{\underset{\beta<\alpha}{\smile} V_{n+1\beta}}(1<\alpha<\tau)$, then $\smile\{N_{n\alpha} \mid n=1,2,\ldots,\alpha<\tau\}=R$. For $x\in V_1$ implies $x\in V_{n1}=N_{n1}$ for some n, and $x\in V_\alpha$, $x\notin V_\beta(\beta<\alpha)$, $1<\alpha<\tau$ imply $x\in V_{n\alpha}$ for some n and $\underset{\beta<\alpha}{\smile} f_\beta(x)\leqq 0$. Since $\underset{\beta<\alpha}{\smile} f_\beta$ is continuous, there exists a nbd (=neighbourhood) $U(x)$ of x such that $U(x)\frown(\underset{\beta<\alpha}{\smile} V_{n+1\beta})=\phi$. Hence $x\notin\overline{\underset{\beta<\alpha}{\smile} V_{n+1\beta}}$, and hence $x\in N_{n\alpha}$.

Next, we shall show $\{N_{n\alpha} \mid \alpha<\tau\}$ is locally finite. Let $V'_\alpha=\left\{x \mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{2}\frac{1}{2^{n+1}}\right\}$, then $V'_\alpha\subseteq V_{n+1\alpha}$. If $x\in V'_\alpha$, $x\notin V'_\beta$ $(\beta<\alpha\leqq\tau)$, then $\underset{\beta<\alpha}{\smile} f_\beta(x)\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2}\frac{1}{2^{n+1}}$. Since $\underset{\beta<\alpha}{\smile} f_\beta$ is continuous, there exists a nbd $V(x)$ of x such that $V(x)\frown V_{n\beta}=\phi(\beta<\alpha)$. Moreover, $x\in V_{n+1\alpha}$ and $V_{n+1\alpha}\frown N_{n\alpha'}=\phi(\alpha'>\alpha)$. Hence there exists a nbd of x intersecting at most one of $N_{n\alpha}(\alpha<\tau)$. Therefore, $F_n=\underset{\alpha}{\smile}\overline{N}_{n\alpha}$ is closed.

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1) α, β, τ denote ordinals in this lemma. In this note covering and refinement mean open covering and open refinement respectively, and notations and terminologies are chiefly due to J. W. Tukey: Convergence and uniformity in topology (1940). The details of the content of this paper will be published in an another place.