2. *Evans's Theorem on Abstract Riemann Surfaces with Null-Boundaries. II*

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(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1956)

Transfinite diameter. Let A be an $\mathfrak{M}$-closed subset of B. We define the transfinite diameter of A of order n as follows

$$\frac{1}{{}_A D_n}=\frac{1}{2\pi {}_nC_2}\Big(\inf \sum_{\substack{p_s, p_t \in A \\ s<t, s, t=1}}^{n,n} G(p_s, p_t)\Big).$$

a) *From the definition, it is clear, if* $A_1 \supseteqq A_2$, ${}_{A_1}D_n \geqq {}_{A_2}D_n$.

b) *Let* Ω *be an ordinary neighbourhood containing* A *with a compact relative boundary. Consider* $1/{}_\Omega D_n=\frac{1}{2\pi}\Big(\frac{1}{{}_nC_2}\inf \sum_{p_s, p_t \in \Omega} G(p_s, p_t)\Big)$. *Then every* p_s *is situated on* $\partial\Omega$.

$\sum_{s<t} G(p_s, p_t)=\sum_{\substack{i,j\neq s \\ i\neq j}} G(p_i, p_j)+\sum_{\substack{i\neq s \\ i=1}}^{n} G(p_s, p_i)$. Then the sum of the first term does not depend on p_s and $\sum_{i\neq s}^{n} G(p_s, p_i)=U(p_s)$ is a superharmonic function of p_s for fixed $\{p_i\}$ in $\overline{R}$. We make $V_M(p_i)$ correspond to every point $p_i\ (i\neq s)$ such that $U(p_s)\geqq M$ in $\bigcup_i V_M(p_i)$, where $M\geqq \max_{p_s\in\partial\Omega} U(p_s)$. Since $U(p_s)$ is $\mathfrak{M}$-lower semicontinuous, $U(p_s)$ attains its minimum m^* at z_0 on an $\mathfrak{M}$-closed set Ω. We show that $z_0 \in \partial\Omega$. If it were not so, assume that $U(z_0)=m^*\leqq m=\min_{p_s\in\partial\Omega} U(p_s)$ in Ω. Suppose $z_0 \in B$, then by 3), $U(z_0)=\frac{1}{2\pi}\int_{\partial V_n(z_0)} U(z)\frac{\partial G(z, z_0)}{\partial n}ds$, where n is so large enough that $V_n(z_0)\subset\Omega$. Then there exists at least one point $r(\in R)$ such that $U(r)\leqq m^*\leqq m$. r must be in $\Omega-\bigcup_i V_M(p_i)$. But since $U(p_s)$ is harmonic non constant in $\Omega-\bigcup_i V_M(p_i)$ and R is a null-boundary Riemann surface, $U(p_s)$ attains its minimum on $\partial\Omega$, by the minimum principle. Thus $U(z_0)>m$ in Ω. This is absurd, therefore every p_i is on $\partial\Omega$.

Let $\omega_\Omega(z)$ be the harmonic measure of Ω with respect to the domain $R-R_0-\Omega$ i.e. $\omega_\Omega(z)$ is harmonic in $R-\Omega-R_0$ and $\omega_\Omega(z)=0$ on ∂R_0, $\omega_\Omega(z)=1$ on $\partial\Omega$.

Since every p_i is on $\partial\Omega$, the following can be proved as in euclidean space,

$$\lim_{n=\infty}\frac{1}{{}_\Omega D_n}=2\pi\Big/\int_{\partial\Omega}\frac{\partial\omega_\Omega(z)}{\partial n}ds=W_\Omega.$$