86. A Theorem on Modules of Trivial Cohomology over a Finite Group

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Let G be a finite group. A (left, say) G-module A is said to be of *trivial cohomology* when the cohomology groups $H^n(H, A)$ vanish for all $n \gtreqless 0$ and for all subgroups H of G. It is known that a G-module A is of trivial cohomology whenever there is an integer r such that $H^r(H, A) = H^{r+1}(H, A) = 0$ for all subgroups H of G. This phenomenon was first noted in Hochschild-Nakayama [5] though for positive-dimensional cohomology groups only (the dimension "lowering" not explicitly given there works also by induction with respect to subgroups) and then independently by Lyndon (unpublished). The detailed proof of the machinery used, fundamental exact sequences, was given in Hochschild-Serre [6]. A simpler proof, making use of cohomological transfer maps, was given by Artin-Tate (Artin-Tate [1], Chevalley [3]). Its significance for Galois cohomology was observed first, by Hochschild-Nakayama [5], in its direct application, in connection of Tsen's theorem for instance, and then by Tate [8] in its application to the cohomology of class field theory, through Artin splitting modules.

Now, in the present note we wish first to note that in case of a p-group G a G-module A is of trivial cohomology as soon as $H^r(G, A) = H^{r+1}(G, A) = 0$ for some integer r; observe that no assumption is made about the cohomology groups on proper subgroups. So, turning to the case of general finite groups, we see that the above theorem may be refined into:

Theorem. *Let A be a G-module. If for every rational prime p (dividing the order $[G]$ of G) there is an integer $r(p)$ such that $H^{r(p)}(H_p, A) = H^{r(p)+1}(H_p, A) = 0$ for a Sylow subgroup H_p of G for p, then A is of trivial cohomology, i.e.*

$$H^n(H, A) = 0$$

for all $n \gtreqless 0$ and for all subgroups H of G.

Leaving details and applications to a subsequent paper, we sketch our proof.

Lemma 1. Let G be a p-group. Let A be a G-module such that $H^r(G, A) = H^{r+1}(G, A) = 0$ for some integer r. Then we have $H^r(G, A \otimes M) = H^{r+1}(G, A \otimes M) = 0$ for any representation module M of G over the ring Z of integers.