100. Contributions to the Theory of Semi-groups. IV

By Kiyoshi ISÉKI
Kobe University

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Following G. Thierrin [7], a semi-group S is called *strongly reversible*, if, for any two elements a, b of S, there are three positive integers r, s and t such that

$$(ab)^r = a^s b^t = b^t a^s.$$

Such a notion is a generalisation of a commutative semi-group.

In this paper, we are mainly concerned with generalisations of the results by S. Schwarz [4–6].

Let $\mathfrak{A}$ be a two-sided ideal of S. We denote by $\overline{\mathfrak{A}}$ the set of element a such that $a^s \in \mathfrak{A}$ for some positive integer s. $\overline{\mathfrak{A}}$ is called the *closure* of $\mathfrak{A}$.

Theorem 1. If a semi-group S is strongly reversible, the closure $\overline{\mathfrak{A}}$ of any two-sided ideal $\mathfrak{A}$ is a two-sided ideal.

Proof. Let a be an element of $\overline{\mathfrak{A}}$ and let x be an element of S. Then there is a positive integer k such that $a^k \in \mathfrak{A}$, and there are three integers r, s and t such that

$$(ax)^r = a^s x^t = x^t a^s.$$

Hence, we have

$$(ax)^{rk} = (a^s x^t)^k = a^{sk} x^{tk} \in \mathfrak{A} x^{tk} \subseteq \mathfrak{A}.$$

Thus $ax \in \overline{\mathfrak{A}}$. Similarly $xa \in \overline{\mathfrak{A}}$. Therefore, $\overline{\mathfrak{A}}$ is a two-sided ideal.

A semi-group S is called a *periodic semi-group*, if, for every element a of S, the semi-group (a) generated by a contains a finite number of different elements.

Such a semi-group has been extensively studied by S. Schwarz.

Theorem 2. Let $\mathfrak{A}$ be a two-sided ideal of a strongly reversible periodic semi-group S, and let $\{e_\alpha\}$ be the set of all idempotents of $\mathfrak{A}$, then

$$\overline{\mathfrak{A}} = \bigcup_\alpha K^{(\alpha)},$$

where $K^{(\alpha)}$ is the largest subsemi-group of S containing only one idempotent e_α.

For the detail of the semi-group $K^{(\alpha)}$, see K. Iséki [3].

Proof. Let $a \in K^{(\alpha)}$, then $a^s = e_\alpha$ for some s. Hence $a \in \overline{\mathfrak{A}}$ and we have $\bigcup_\alpha K^{(\alpha)} \subseteq \overline{\mathfrak{A}}$. Conversely, let $a \in \overline{\mathfrak{A}}$, then $a^s \in \mathfrak{A}$ for some s. Hence there is an integer t such that $(a^s)^t = e_\alpha \in K^{(\alpha)}$. This shows $a \in K^{(\alpha)}$.