125. On Closed Mappings

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1. Introduction. In a previous paper [6], S. Hanai and the author have dealt with the problem: "Under what condition will the image of a metric space under a closed continuous mapping be metrizable ?", and obtained the second part of the following theorem; this result, as M. Tsuda has called our attention, was also obtained by A. H. Stone and announced in [7].

Theorem 1. *Let X be a metric space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is paracompact and perfectly normal. Furthermore, Y is metrizable if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y.*

In the present note we shall deduce the first part of Theorem 1 as an immediate consequence of Theorem 3 below, and establish an analogous result for the case of locally compact spaces; namely we shall prove the following theorems.

Theorem 2. *Let X be a paracompact and locally compact Hausdorff space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is a paracompact Hausdorff space. Furthermore Y is locally compact if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y.*

Theorem 3. *Let X be a paracompact and perfectly normal space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is paracompact and perfectly normal.*

The second part of Theorem 2 is a direct consequence of Theorem 4 below.

Theorem 4. *Let f be a closed continuous mapping of a paracompact and locally compact Hausdorff space X onto another topological space Y. Denote by Y_0 [or Y_1] the set of all points y of Y such that $f^{-1}(y)$ [or $\mathfrak{B}f^{-1}(y)$] is not compact. Then we have $Y_1 \subset Y_0$ and*

(a) *Y_0 is a closed discrete subset of Y;*

(b) *$Y-Y_1$ is locally compact;*

(c) *the closure of any neighbourhood of y is not compact for every point y of Y_1.*

From Theorem 4 we obtain immediately

Corollary. *Under the assumption of Theorem 4 the mapping f admits of a factorization $f=f_2\circ f_1$ such that*