167. A Remark on Fundamental Exact Sequences in Cohomology of Finite Groups

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The purpose of the present short note is to establish two exact sequences, and their duals, which form a bridge between the well-known series of so-called fundamental exact sequences in cohomology and homology of finite groups, and one of which has been made use of, if not in a way of absolute necessity, in a recent note [7] by the writer. Thus, we prove: Let G be a finite group, H an invariant subgroup of G, and M a G-module. Then the sequence

$$(2'_0)\qquad 0\longleftarrow H^0(G/H, M^H)\xleftarrow{\varphi'} H^0(G,M)\xleftarrow{\iota} H^0(H,M)_G \xleftarrow{\tau'} H^{-1}(G/H, M^H)\xleftarrow{\varphi'} H^{-1}(G,M)$$

is exact.[1] Further, if $H^0(H,M)=0$, then the sequence

$$(2'_1)\qquad 0\longleftarrow H^{-1}(G/H, M^H)\xleftarrow{\varphi'} H^{-1}(G,M)\xleftarrow{\iota} H^{-1}(H,M)_G \xleftarrow{\tau'} H^{-2}(G/H, M)\xleftarrow{\varphi'} H^{-2}(G,M)$$

is exact.[2] Dually, the sequence

$$(1'_{-1})\qquad 0\longrightarrow H^{-1}(G/H, M_H)\xrightarrow{\lambda'} H^{-1}(G,M)\xrightarrow{\rho} H^{-1}(H,M)^G \xrightarrow{\tau'} H^0(G/H, M_H)\xrightarrow{\lambda'} H^0(G,M)$$

is exact. If $H^{-1}(H,M)=0$, then the sequence

$$(1')\qquad 0\longrightarrow H^0(G/H, M_H)\xrightarrow{\lambda'} H^0(G,M)\xrightarrow{\rho} H^0(H,M)^G \xrightarrow{\tau'} H^1(G/H, M_H)\xrightarrow{\lambda'} H^1(G,M)$$

is exact. The significance of the maps in these sequences will be given in the sequel.

To begin with, we consider a not necessarily finite group G and an invariant subgroup H of G. Let M be a G-module. Then (Hochschild-Nakayama [5], Hochschild-Serre [6]):

I. If $m\geqq 1$ and if $H^n(H,M)=0$ for $n=1,2,\cdots, m-1$, then the sequence of cohomology groups

$$(1)\qquad 0\longrightarrow H^m(G/H, M^H)\xrightarrow{\lambda} H^m(G,M)\xrightarrow{\rho} H^m(H,M) \xrightarrow{\tau} H^{m+1}(G/H, M^H)\xrightarrow{\lambda} H^{m+1}(G,M)$$

is exact, where λ is a lifting map, ρ is a restriction map, and τ is a so-called transgression; that the transgression maps precisely

1) The first half of this exact sequence has been given in Artin-Tate [1].

2) The first half of this exact sequence has independently been obtained by Y. Kawada.