5. Some Properties of Completely Normal Spaces

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1. Recently, T. Inokuma [1] has given an answer to a problem proposed by A. D. Wallace [2] by proving that a topological space satisfies the condition of Wallace if and only if it is completely normal. Moreover, he has proved the following theorem:

If a topological space H is completely normal and fully normal, then for any locally finite family of subsets $X_1, X_2, \cdots$ of H, there exists a family of closed sets $H_1, H_2, \cdots$ satisfying the condition (V*):

$$(\mathrm{V}^*)\qquad \begin{cases} H=H_1\cup H_2\cup\cdots \\ H_i\cap H_j\cap(\overline{X}_i\cup\overline{X}_j)=\overline{X}_i\cap\overline{X}_j \\ \overline{X}_i\subset H_i \qquad i=1,2,\cdots . \end{cases}$$

In this paper we shall prove that in the above theorem the assumption of "full normality" is superfluous, and moreover, we shall establish some other related properties of completely normal spaces.

2. We shall first remark that the third relation of (V*) is derived from the second one; this is seen by putting $i=j$ in the second relation of (V*).

Secondly, without loss of generality we may assume that every X_i in the above theorem is a closed subset of H; so each X_i is assumed to be a closed subset of H in the following.

We can easily prove the following lemma by a simple calculation of sets.

Lemma 1. *Let H be a set, and let $\{X_\alpha|\alpha\in\Omega\}$ and $\{H_\alpha|\alpha\in\Omega\}$ be two families of subsets of the set H; then the following three conditions* (A), (B) *and* (C) *are equivalent*:

$$(\mathrm{A})\qquad \begin{cases} (\mathrm{A1})\quad H=\bigcup_{\alpha\in\Omega} H_\alpha \\ (\mathrm{A2})\quad H_\alpha\cap H_\beta\cap(X_\alpha\cup X_\beta)=X_\alpha\cap X_\beta, \qquad (\alpha,\beta\in\Omega) \end{cases}$$

$$(\mathrm{B})\qquad \begin{cases} (\mathrm{B1})=(\mathrm{A1}) \\ (\mathrm{B2})\quad H_\alpha\cap X_\beta=X_\alpha\cap X_\beta, \qquad (\alpha,\beta\in\Omega) \end{cases}$$

$$(\mathrm{C})\qquad \begin{cases} (\mathrm{C1})=(\mathrm{A1}) \\ (\mathrm{C2})\quad H_\alpha\cap(\bigcup_{\gamma\in\Omega} X_\gamma)=X_\alpha, \qquad (\alpha\in\Omega). \end{cases}$$

Since the condition (V*) is, of course, identical with the condition (A) of Lemma 1, we obtain the following theorem by T. Inokuma [1, Theorems 1, 2].

Theorem 1. *In order that a topological space H be a completely normal space, it is necessary and sufficient that for any finite family*