2. *Fourier Series. VI. A Convergence Theorem*

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1. We have proved the following theorem [1].

Theorem 1. *If, for a fixed* x

$$\int_0^t [f(x+u)-f(x)]du=o(t) \qquad as \quad t\to 0 \tag{1}$$

and

$$\int_0^t [f(\xi+u)-f(\xi-u)]du=o\left(t/\log\frac{1}{t}\right) \qquad as \quad t\to 0 \tag{2}$$

uniformly in ξ *in a neighbourhood of* x, *the Fourier series of* $f(t)$ *converges at* $t=x$.

Further S. Izumi [2] has proved

Theorem 2. *If, for a fixed* x

$$\int_0^t |f(x+u)-f(x)|du=o(t) \qquad as \quad t\to 0 \tag{3}$$

and

$$n\int_0^{\pi/n} dt\left|\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}\frac{f(v)-f(v-\pi/n)}{v}dv\right|=o(1) \tag{4}$$

as $n\to\infty$, *then the Fourier series of* $f(t)$ *converges at* $t=x$.

We shall here prove the following theorems.

Theorem 3. *If the Fourier series of* $f(t)$ *is summable* $(C,1)$ *at* $t=x$ *and the condition (2) holds, then the Fourier series of* $f(t)$ *converges at* $t=x$.

Theorem 4. *If the Fourier series of* $f(t)$ *is summable* $(C,1)$ *at* $t=x$ *and*

$$\int_0^t [f(\xi+u)-f(\xi-u)]du=o(t) \qquad as \quad t\to 0 \tag{5}$$

uniformly in ξ *in a neighbourhood of* x, *and further if**)

$$\int_0^{\pi/n}\left|\sum_{\substack{j=-[n/2]\\ j\neq 0}}^{[n/2]}\frac{\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)}{t+2j\pi/n}\right|dt=o(t) \qquad as \quad n\to\infty, \tag{6}$$

then the Fourier series of $f(t)$ *converges at* $t=x$.

It is known that the condition (3) implies (C, 1) summability of Fourier series of $f(t)$ at $t=x$, but the condition (1) does not so.

For the proof of these theorems we make use of the following theorem, due to W. W. Rogosinski [3]:

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*) Δ^2_h is the second difference with breadth h, and then $\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n) =f(x+t+(2j-1)\pi/n)-2f(x+t+2j\pi/n)+f(x+t+(2j+1)\pi/n)$.