1. Fourier Series. V. A Divergence Theorem

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1. In the Mathematical Reviews [1], the following theorem is reviewed.[1]

Theorem A. *If $f(x)$ is integrable and $f(x)=0$ in a closed set E in $(-\pi, \pi)$, then the Fourier series of $f(x)$ converges to zero in each density point of E, when*

$$(1) \qquad \sum_{k=1}^{\infty} \omega(\delta_k, f) < \infty,$$

(δ_k) being intervals contiguous to E and $\omega(\delta, f)$ denoting the oscillation of f in the interval δ.

K. Tandori [2] proved that, in the above theorem, the condition (1) can not be omitted; that is, there are a closed set E and a continuous function $f(x)$ such that $f(x)=0$ in E, $x=0$ is the density point of E and the Fourier series of $f(x)$ diverges at $x=0$.

We shall here prove that Theorem A is false,[2] that is,

Theorem 1. *There are a closed set E of positive measure, with $x=0$ as a density point and an integrable function $f(x)$ such that $f(x)=0$ in E and the Fourier series of $f(x)$ diverges at $x=0$ and that the condition (1) is satisfied.*

But Theorem A holds true when the integrability of $f(x)$ is replaced by its continuity. More generally,

Theorem 2. *If $f(x)$ is an integrable function such that $f(x)=0$ in a closed set E in $(-\pi, \pi)$, and*

$$(2) \qquad \sum_{k=1}^{\infty} \omega(\bar{\delta}_k, f) < \infty,$$

where $\bar{\delta}_k$ denotes the closure of δ_k, a contiguous interval of E. Then the Fourier series of $f(x)$ converges to zero in each density point of E.

2. **Proof of Theorem 1.** Let (n_k) be an increasing sequence of integers such that $n_k > n_{k-1}^2$. Let (δ_j) be a sequence of open intervals such that

$$\delta_j \subset (\pi/(2j+1), \pi/2j) \qquad (j=n_k, n_k+1, \cdots, n_k^2)$$

and the length of δ_j is $O(1/j^3)$. We put $E=(-\pi, \pi)-\vee \delta_j$. We define $f(x)$ such that

1) The author could not refer the original paper.

2) We consider $\omega(\delta_k, f)$ as the oscillation of the open interval δ_k