31. *Divergent Integrals as Viewed from the Theory of Functional Analysis. II**)

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§ 6. *The examination of analyticity.*

We can see after the integration by part that if $v(k, s)$ is an analytic function of k, v^* satisfies $\frac{\partial}{\partial k} v^*=0$ $\left(\frac{\partial}{\partial k}=\frac{1}{2}\left(\frac{\partial}{\partial \sigma}+i\frac{\partial}{\partial \tau}\right)\right)$, and $\Delta v^*=0$ $\left(\Delta=\frac{\partial^2}{\partial\sigma^2}+\frac{\partial^2}{\partial\tau^2}\right)$

However in our space Φ' either the equation $\frac{\partial}{\partial k} v^*=0$ or $\Delta v^*=0$ can not be a criterion of the analyticity of v^* unlikely to the case in $\mathfrak{D}'$. We see this fact easily from the following counter example. If $v\equiv 1$, both equations hold for v^*, but v^* is not regular at the origin (Example 2).

As already seen in § 3, no function $\varphi(\sigma, \tau)$ of Φ has a compact carrier. However we saw also in § 3 that any element φ of $\mathfrak{D}_L(\sigma, \tau)$ can be approximated by $\{\varphi_j \mid \varphi_j \in \Phi\}$ in the topology S. Hence we can see that when $v(k, s)$ is an analytic function of k, v^* is equivalent in Φ' to an analytic function on a compact set $L(\subset D_1)$ if v^* is continuous for such sequence $\{\varphi_j \mid \varphi_j \underset{S}{\longrightarrow} \varphi,\ \varphi \in \mathfrak{D}_L(\sigma, \tau),\ \varphi_j \in \Phi\}$.

In the following we see three examples of our divergent integrals which are the Laplace transforms. Example 1 has no singularity on its abscissa of convergence. Example 2 has one singular point on its abscissa of convergence, and Example 3 has its natural boundary on its abscissa of convergence.

Example 1. $f(s)=\int_0^\infty e^{-st}F(t)dt$ where $F(t)=-\pi e^t \sin(\pi e^t)$. This integral diverges on $R(s)\leq 0$, and $\mathfrak{L}^{(k)}$-transform (by Cesàro's methods of summation of order k) is convergent on $R(s)>-k$ for arbitrary k [2].

We consider this integral as above, for example for the case $k=2$. We take the domain $-2+\varepsilon\leq\tau\leq\tau_2<\infty$, $-\infty<\sigma<+\infty$, as D_1. By repeated partial integration we see

$$f(s,t)=\int_0^t e^{-st}F(t)\,dt=1+e^{-st}\cos(\pi e^t)+\frac{s}{\pi}e^{-(s+1)t}\sin(\pi e^t)$$
$$-\frac{s(s+1)}{\pi^2}-\frac{s(s+1)}{\pi^2}e^{-(s+2)t}\cos(\pi e^t)$$

*) T. Ishihara [1].