73. Fourier Series. XVI. The Gibbs Phenomenon of Partial Sums and Cesàro Means of Fourier Series. 2

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5. Proof of Theorem 7. Let

$$n_k = 2^{2^k} \quad (k=1,2,\cdots).$$

Then $\quad 2\sqrt{n_k}\,\pi/n_k = 2\pi/\sqrt{n_k} = 2\pi/2^{2^{k-1}} = 2\pi/n_{k-1}$.

Let $\varphi_k(x)$ be an even concave function which is zero for $x \geqq \pi/2n_k$ and such that its curve touches y-axis at $y=1$ and touches x-axis at $x=\pi/2n_k$. Further suppose[1]

$$\int_0^t \varphi_k(x)\,dx - t\varphi_k(t) = t \Big/ \sqrt{\log\log\frac{1}{t}}$$

for all $0 < t \leqq \pi/2n_k$.

Let

$$\begin{aligned} f_k(x) &= \varphi_k(x+(2j-1/2)\pi/n_k) \quad && \text{in } ((2j-1)\pi/n_k,\ 2j\pi/n_k), \\ &= 0 && \text{otherwise}, \end{aligned}$$
$$(j=\sqrt{n_k}/\log n_k,\ (\sqrt{n_k}/\log n_k)+1,\cdots,\sqrt{n_k}),$$

and

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Then

$$s_{n_k}(\pi/n_k, f) = s_{n_k}(\pi/n_k, f_k) + o(1).$$

If we set $\psi_k(t) = \varphi_k(t+\pi/2n_k)$, then

$$\begin{aligned} s_{n_k}(\pi/n_k, f_k) &= \frac{1}{\pi}\int_0^{\pi} f_k(t+\pi/n_k)\frac{\sin n_k t}{t}\,dt + o(1) \\ &= \frac{1}{\pi}\sum_{j=\sqrt{n_k}/\log n_k}^{\sqrt{n_k}} \int_0^{\pi/n_k} \psi_k(t)\frac{\sin n_k t}{t+2j\pi/n_k}\,dt + o(1) \\ &\geqq \frac{1}{\pi}\int_0^{\pi/n_k}\psi_k(t)\sin n_k t\,dt \sum_{j=\sqrt{n_k}/\log n_k}^{\sqrt{n_k}} \frac{n_k}{2j\pi} + o(1) \\ &\geqq A\log\log n_k\cdot n_k\int_0^{\pi/n_k}\psi_k(t)\sin n_k t\,dt + o(1) \\ &\geqq A\log\log n_k\cdot n_k\int_{\pi/4n_k}^{3\pi/4n_k}\psi_k(t)\,dt + o(1) \\ &\geqq A\log\log n_k/\sqrt{\log\log n_k}. \end{aligned}$$

Hence $s_{n_k}(\pi/n_k, f) \to \infty$ as $k\to\infty$. Thus partial sums of Fourier series of $f(x)$ present the Gibbs phenomenon at $x=0$.

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1) The base of logarithm is 2.