152. A Theorem on Residuated Lattices

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1. Let L be a complete lattice-ordered semigroup (cl-semigroup) with a maximally integral identity[1] e, and suppose that L has a unique mapping into itself $a\to a^{-1}$ with two properties 1) $aa^{-1}a\leq a$ and 2) $axa\leq a$ implies $x\leq a^{-1}$. In the previous paper [1], we obtained[2] that *L forms a commutative cl-group which is a direct product of infinite cyclic groups generated by prime elements, if L satisfies the following conditions*:

(1) *The ascending chain condition (a.c.c.) holds for integral elements of L.*

(2) *Any prime element is divisor-free (maximal).*

(3) *Any prime element contains an element c satisfying $c^{-1^{-1}}=c$.*

Our purpose of the present note is to show that the condition (1) is replaceable equivalently by *the restricted descending chain condition for integral elements of L.*

2. Let L be a cl-semigroup with an identity e. If e is maximally integral, then, in order that L has a mapping into itself $a\to a^{-1}$ with above two properties 1) and 2), it is necessary and sufficient that L forms a residuated lattice.[3] In [1] we have proved[4] that the condition is necessary. We show that the condition is sufficient. Suppose that L is a residuated lattice. Then $(e:a)_l=(e:a)_r$. For, let $ax\leq e$, then $xaxa\leq xa$, $(xa\smile e)^2\leq xa\smile e$, and so $xa\smile e=e$, $xa\leq e$. Hence $(e:a)_l\leq(e:a)_r$. Similarly $(e:a)_r\leq(e:a)_l$. We get therefore $(e:a)_l=(e:a)_r$. We next prove that $e=(a:a)_l=(a:a)_r$. Since $(a:a)_ra\leq a$, we have $(a:a)_r^2a\leq a$, $(a:a)_r^2\leq(a:a)_r$. $(a:a)_r\geq e$ is evident. Hence $e=(a:a)_r$, similarly $e=(a:a)_l$. We now define a mapping $a\to a^{-1}$ with $a^{-1}=(e:a)_l=(e:a)_r$. Then $aa^{-1}a=a\cdot(e:a)_ra\leq ae=a$, and $axa\leq a$ implies $ax\leq(a:a)_r=e$, hence $x\leq(e:a)_l=a^{-1}$.

Lemma 1. *Let a and b be two elements in L. If b covers a, then $(a:b)_l$ is a prime element. In particular, if b is integral, then $(a:b)_l$ is a prime element containing a. Similarly for $(a:b)_r$.*

Proof. Suppose that $bx\leq a$. Then $abx\leq a^2\leq ab$. Hence $x\leq(ab:ab)_l$

1) An element x is called *integral* if $x^2\leq x$. e is called *maximally integral* if $e\leq c$ ($c^2\leq c$) implies $e=c$.

2) Cf. [1, p. 14, Theorem 2.6].

3) Cf. [2, p. 201]. $(a:b)_l$ will denote the left residual of a by b which is the largest x satisfying $bx\leq a$. Symmetrically for the right residual $(a:b)_r$ of a by b.

4) Cf. [1, p. 12, Theorem 2.2].