146. Note on a Theorem for Metrizability

By Jun-iti NAGATA
Department of Mathematics, Osaka City University

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In the present note, we shall apply the previous metrization theorem[1] to an open problem and shall prove the metrizability of a T_1-space X satisfying the following condition of T. Inagaki:[2]

For every point p of X, we can assign a nbd (=neighborhood) basis $\{V_n(p) \mid n=1,2\cdots\}$ such that

I) *for every $p \in X$ and n, there exists $m=\alpha(p,n)$ such that $p \in V_m(q)$ implies $V_m(q) \subseteq V_n(p)$,*

II) *for every $p \in X$ and n, there exists $l=\beta(p,n)$ such that $q \in V_l(p)$ implies $p \in V_n(q)$.*

Theorem. *In order that a T_1-space X is metrizable it is necessary and sufficient that X satisfies the above condition.*

Proof. Since the necessity is clear, we prove only the sufficiency.

1. First, we remark that we can assume, without loss of generality, that $m<n$ implies $V_m(p) \supseteq V_n(p)$ for every $p \in X$; otherwise we have the fulfilment of the condition by replacing $V_n(p)$ with $V_1(p) \frown \cdots \frown V_n(p)$.

2. For every $p \in X$ and n, we can choose $k=\gamma(p,n)$ such that $q \in V_k(p)$ implies $p \in V_m(q) \subseteq V_n(p)$ for $m=\alpha(p,n)$.

To show this, let $m=\alpha(p,n)$, $k=\beta(p,m)=\gamma(p,n)$. Then $q \in V_k(p)$ implies $p \in V_m(q) \subseteq V_n(p)$ by I) and II).

3. For every $p \in X$ and n, there exist nbds $M_n^1(p)$ and $M_n^2(p)$ of p such that $q \notin V_n(p)$ implies $M_n^1(p) \frown M_n^2(q)=\phi$.

We let $k=\gamma(p,n),\quad l=\beta(p,n),\quad k'=\gamma(p,l);$

$$V_k(p)=M_n^1(p),\qquad V_{k'}(p)=M_n^2(p).$$

Now, let $q \notin V_n(p)$, $r \in M_n^1(p) \frown M_n^2(q) \neq \phi$.

Then in the case of $m=\alpha(p,n) \leqq \alpha(q,l)=m'$,[3] we have

$$q \in V_{m'}(r) \subseteq V_m(r) \subseteq V_n(p)$$

from 2, which contradicts $q \notin V_n(p)$.

In the case of $m=\alpha(p,n) \geqq \alpha(q,l)=m'$, we have

$$p \in V_m(r) \subseteq V_{m'}(r) \subseteq V_l(q),$$

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1) J. Nagata: A theorem for metrizability of a topological space, Proc. Japan Acad., **33**, no. 3 (1957), Theorem 1. See, also, J. Nagata: A contribution to the theory of metrization, Jour. Inst. Polytech., Osaka City Univ., **8**, no. 2 (1957).

2) T. Inagaki: Sur les espaces à structure uniforme, Jour. of the Faculty of Sciences, Hokkaido University, **10** (1943). Prof. Inagaki proved in the paper that a separable space satisfying this condition was perfectly separable. We have learned from Prof. K. Morita that the metrization of such a space is an open problem.

3) We remark that this l does not mean $\beta(p,n)$ but $\beta(q,n)$.