

97. A Characterisation of P -spaces

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(Comm. by K. KUNUGI, M.J.A., July 12, 1958)

The concept of P -spaces was introduced by L. Gillman and M. Henriksen [1]. Following their definitions, we shall define P -point and P -space.

A point x of a topological space X is called a P -point if every countable intersection of neighborhoods of x contains a neighborhood of x . X is called P -space, if every point of X is a P -point.

In my papers published in Proc. Japan Acad. (1957–1958), some topological spaces have been characterised by *the properties on sequences of continuous functions*. In this Note, we shall give a characterisation of P -space by a similar consideration. To do it, we suppose that a topological space X is completely regular. Then we have the following

Theorem. *A completely regular space is P -space, if and only if the limit function of any convergent sequence of continuous functions is continuous.*

Proof. Let $\{f_n(x)\}$ be a convergent sequence of continuous functions $f_n(x)$ on a completely regular P -space, and let $f(x)$ be its limit. To prove that $f(x)$ is continuous on X , for a point a of X and a given positive ε , we take the neighborhood $U_n(a) = \{x \mid |f_n(a) - f_n(x)| < \varepsilon\}$ of a . Since X is P -space, $U = \bigcap_{n=1}^{\infty} U_n(a)$ is a neighborhood of a , and we have

$$|f_n(a) - f_n(x)| < \varepsilon \quad \text{for } x \in U \quad (n=1, 2, \dots).$$

Therefore, for $n \rightarrow \infty$, we have

$$|f(a) - f(x)| \leq \varepsilon \quad \text{for } x \in U.$$

This shows that $f(x)$ is continuous at $a \in X$.

Conversely, suppose that there is a sequence of neighborhoods U_n of a such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of a . Since X is completely regular, for every n , we can find a continuous function $f_n(x)$ such that $0 \leq f_n(x) \leq 1$ and

$$f_n(x) = \begin{cases} 0 & x \notin U_n, \\ 1 & x = a. \end{cases}$$

Let $g_n(x) = \text{Min}\{f_1(x), f_2(x), \dots, f_n(x)\}$, then the sequence $\{g_n(x)\}$ is decreasing and $0 \leq g_n(x) \leq 1$. Therefore $\{g_n(x)\}$ is convergent to a function $g(x)$ on X . Since the intersection $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of a , for a given neighborhood U of a , we can take a point b such that $U - \bigcap_{n=1}^{\infty} U_n \ni b$. Hence $b \notin \bigcap_{n=1}^{\infty} U_n$ and we can find a neighborhood U_{n_0} such