134. On the Structure of the Associated Modular

By Tsuyoshi ANDÔ
Research Institute of Applied Electricity, Hokkaidô University
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Let R be a modulared semi-ordered linear space[1] with a modular m. The structure of the conjugate modular $\overline{m}$ on the conjugate space $\overline{R}^m$ is investigated in detail.[2] On the other hand, it is known[3] that if the norm on R by m is not continuous, $\overline{R}^m$ constitutes a *proper* normal manifold of the associated space $\widetilde{R}^m$. In this short note, we shall determine completely the structure of the associated modular $\widetilde{m}$ on the orthogonal complement $(\overline{R}^m)^\perp$ of $\overline{R}^m$ in $\widetilde{R}^m$.

Theorem. *The associated modular* $\widetilde{m}$ *is linear*[4] *on* $(\overline{R}^m)^\perp$; *more precisely it is given by the formula*:

$$\widetilde{m}(\widetilde{a}) = \sup_{m(x)<\infty} |\widetilde{a}(x)| \qquad \textit{for all } \widetilde{a}\in(\overline{R}^m)^\perp.$$

Proof. There exists[5] a normal manifold N of R such that m is semi-simple on N and is singular on $N^\perp$. It is known that N is semi-regular[6] and the associated modular $\widetilde{m}$ is linear on $[N^\perp]\widetilde{R}^m$. Thus to prove Theorem we may assume that R is semi-regular.

Let $0\leqq\widetilde{a}\in(\overline{R}^m)^\perp$ and $0\leqq a\in R \quad m(a)<\infty$.

Put $$\boldsymbol{F}=\{x;\ 0\leqq x\leqq a \quad \widetilde{a}(x)=0\}.$$

Then it is a lattice manifold. Putting $e=\bigcup_{x\in \boldsymbol{F}} x$, we shall show first that $a=e$. For this purpose, it is sufficient to prove that

$$\overline{x}(a-e)\leqq\varepsilon \qquad \text{for any } 0\leqq\overline{x}\in\overline{R}^m \text{ and } \varepsilon>0,$$

because R is semi-regular by assumption. Since $\widetilde{a}\frown\overline{x}=0$, there exist[7] $\{b_\nu\}_{\nu=1}^\infty\subset R$ such that

$$0\leqq b_\nu\leqq a \quad \text{and} \quad \overline{x}(a-b_\nu)+\widetilde{a}(b_\nu)\leqq\varepsilon/2^\nu \quad (\nu=1,2,\cdots).$$

Putting $b=\bigcap_{\nu=1}^\infty b_\nu$, we have $0\leqq\widetilde{a}(b)\leqq\inf_{\nu=1,2,\cdots}\widetilde{a}(b_\nu)=0$, namely $b\in\boldsymbol{F}$. Further universal continuity of $\overline{x}$ implies $\overline{x}(a-b)=\overline{x}(\bigcup_{\nu=1}^\infty(a-b_\nu))\leqq\sum_{\nu=1}^\infty\overline{x}(a-b_\nu)\leqq\varepsilon$. From this and the definition of e it follows that

1) We use the definitions, terminology, and notations in H. Nakano: Modulared Semi-ordered Linear Spaces, Maruzen, Tokyo (1950).
2) Ibid., §§ 41–46.
3) Ibid., Theorem 31.10.
4) $\widetilde{m}(\xi\widetilde{a})=\xi\widetilde{m}(\widetilde{a})$ for all $\xi\geqq0$.
5) Ibid., § 35.
6) Semi-regularity means that $\overline{x}(a)=0$ (for all $\overline{x}\in\overline{R}^m$) implies $a=0$.
7) Ibid., § 18.