130. *On Linear Functionals of W*-algebras*

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1958)

1. We shall explain the background of our study.

Let B be the W^*-algebra of all bounded operators on a Hilbert space H, then σ-weakly continuous linear functionals on B are identified with operators of trace class u in H as follows: $\psi_u(a)=\mathrm{Tr}(ua)$ $(a\in B)$. Self-adjoint (resp. positive) operators u of trace class correspond exactly to σ-weakly continuous self-adjoint (resp. positive) linear functionals ψ_u and the trace-norm $\|u\|_1=\mathrm{Tr}((u^*u)^{1/2})$ of u is equal to the norm $\|\psi_u\|$ of corresponding functionals. If u is self-adjoint, it can be written under $u=v-w$, where v and w are its positive and negative parts, and $\|u\|_1=\|v\|_1+\|w\|_1$. Besides, if we have $u=v'-w'$, where $v', w'\geqq 0$ and $\|u\|_1=\|v'\|_1+\|w'\|_1$, then we can easily show that $v=v'$ and $w=w'$. Namely: A σ-weakly continuous self-adjoint functional ψ_u on B can be written under $\psi_u=\psi_v-\psi_w$, where $\psi_v, \psi_w\geqq 0$ such that $\|\psi_u\|=\|\psi_v\|+\|\psi_w\|$, and such decomposition is unique. Grothendieck [3] has shown that this fact holds also valid in general W^*-algebras.

On the other hand, we know a stronger fact in B as follows: Let t be an operator of trace class, $t=v|t|$ $(|t|=(t^*t)^{1/2})$ its polar decomposition, then $\|t\|_1=\|\,|t|\,\|_1$ and v is a partially isometric operator $(\in B)$ having the range projection of $|t|$ as the initial projection. Now we consider the functional ψ_t, and denote $\psi_t(xy)=\hat{Y}\psi_t(x)$ and $\psi_t(yx)=\hat{\hat{Y}}\psi_t(x)$ for $x, y\in B$, then since $\psi_t(xy)=\mathrm{Tr}(txy)=\mathrm{Tr}(ytx)$, the above fact implies: $\psi_t=\hat{V}\psi_{|t|}$, $\|\psi_t\|=\|\psi_{|t|}\|$ and $\hat{V}$ is a partially isometric operator having the support $S(\psi_{|t|})$ of $\psi_{|t|}$ as the initial projection, where for $\psi\geqq 0$, $S(\psi)=I-\sup e$ [e, projections such that $\psi(e)=0$].

Moreover we can easily show that such decomposition is unique, and call this decomposition *the polar decomposition of functionals.*

Our purpose of this note is to show that the polar decomposition of functionals is also valid in general W^*-algebras.

2. We shall state

Theorem 1. *Suppose a W^*-algebra M realized as a W^*-subalgebra of the algebra B on a Hilbert space H, then a σ-weakly continuous linear functional ψ on M is the restriction of a σ-weakly continuous linear functional of the same norm on B.*

Proof. It is enough to suppose $\|\psi\|=1$. Let S be the unit sphere of M and $F=\{a\mid |\psi(a)|=1,\ a\in S\}$, then F is a non-void, convex, σ-