154. Note on Idempotent Semigroups. V. Implications of Two Variables

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§ 1. This note is the continuation of the previous papers (Kimura [1, 3, 4]; Yamada and Kimura [2]). Any terminology without definition should be referred to them.

The purpose of this note is to present the classification of all implications of two variables on idempotent semigroups and some relevant matters.

The proofs of any lemmas and theorems are all omitted, which will be given in detail elsewhere.[1]

Let $X=\{x_1, x_2,\cdots,x_n\}$ be a set where each element x_i is called a variable. Let S be an idempotent semigroup. Then, we call every element $x_{i1}x_{i2}\cdots x_{im}$ of the free semigroup generated by X a *polynomial* of n variables $x_1,x_2,\cdots,x_n$ on S when we regard each element of X as a variable on S and $x_{i1}x_{i2}\cdots x_{im}$ as the product of $x_{i1}x_{i2}\cdots x_{im}$ with respect to the multiplication in S, and denote by $f(x_1, x_2,\cdots, x_n)$, etc.

Take up four families of polynomials, $\{f_i: i\in I\}$, $\{g_i: i\in I\}$, $\{f_j^*: j\in J\}$ and $\{g_j^*: j\in J\}$, of the same variables $x_1, x_2,\cdots, x_n$, and consider a relation defined by

(P.I) $$\{f_i(x_1, x_2,\cdots, x_n)=g_i(x_1, x_2,\cdots, x_n) : i\in I\} \quad \text{implies}$$
$$\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}.$$

Such a relation is called a *polynomial implication*, or more simply an *implication*, of n variables $x_1, x_2,\cdots, x_n$ on idempotent semigroups, and denoted by

$$\{f_i(x_1, x_2,\cdots, x_n)=g_i(x_1, x_2,\cdots, x_n) : i\in I\} \Rightarrow$$
$$\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}.$$

For example, $\{xy=yx, xyx=yx\}\Rightarrow x=y$ is an implication of two variables x, y. Particularly, an implication is called "*trivial*" if it's satisfied by all idempotent semigroups. Further, if $\{f_i(x_1, x_2,\cdots, x_n)=g_i(x_1, x_2,\cdots, x_n) : i\in I\}$ in (P.I) consists of only trivial identities[2] on idempotent semigroups, then any idempotent semigroup satisfies (P.I) if and only if it satisfies identities $\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}$, and therefore in this case (P.I) is called especially a *family of identities*.

1) This is an abstract of the paper which will appear elsewhere.

2) An equality $f(x_1, x_2,\cdots, x_n)=g(x_1, x_2,\cdots, x_n)$ is called a trivial identity on idempotent semigroups if it becomes an identity for all idempotent semigroups.