13. *On Fejér Kernels*

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This note is a selection slightly modified of the parts concerning the properties of Fejér kernels and their conjugates studied in Yano [1-3], the paper [1] being written in Japanese. The results will improve the lemmas used in Gergen [4] and others.

1. Fejér kernels. The results in this article have been known classically in alternative forms. Here we shall deal with Fejér kernels and their conjugates at the same time. The n-th Fejér kernel of order α, $\alpha>-1$, is

$$\mathrm{K}_n^\alpha(t)=\frac{1}{2}+\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^\alpha \cos \nu t, \tag{1.1}$$

where A_n^γ, $-\infty<\gamma<\infty$, is defined by the identity

$$(1-x)^{-\gamma-1}=\sum_{n=0}^{\infty}A_n^\gamma x^n \quad (|x|<1), \tag{1.2}^{*)}$$

and its conjugate is

$$\overline{\mathrm{K}}_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^\alpha \sin \nu t. \tag{1.3}$$

Putting

$$g_n^\alpha(t)=\mathrm{K}_n^\alpha(t)+i\overline{\mathrm{K}}_n^\alpha(t), \tag{1.4}$$

we have by (1.1) and (1.3)

$$g_n^\alpha(t)=-\frac{1}{2}+\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^\alpha e^{i\nu t}=-\frac{1}{2}+\frac{e^{int}}{A_n^\alpha}\sum_{\nu=0}^{n}A_\nu^\alpha e^{-i\nu t}.$$

Applying Abel's transformation to the last sum $\sum_{\nu=0}^{n}$ once,

$$g_n^\alpha(t)=\frac{i}{2}\cot\frac{1}{2}t+\frac{e^{int}}{A_n^\alpha(1-e^{-it})}\sum_{\nu=0}^{n}A_\nu^{\alpha-1}e^{-i\nu t}.$$

Generally, applying Abel's transformation to the last sum m-times successively we get

$$\begin{aligned} g_n^\alpha(t)=\frac{i}{2}\cot\frac{1}{2}t-\sum_{j=1}^{m}\frac{A_n^{\alpha-j}e^{-it}}{A_n^\alpha(1-e^{-it})^{j+1}} \\ +\frac{e^{int}}{A_n^\alpha(1-e^{-it})^{m+1}}\sum_{\nu=0}^{n}A_\nu^{\alpha-m-1}e^{-i\nu t}. \end{aligned} \tag{1.5}$$

When $m\geqq\alpha$ the series $\sum A_\nu^{\alpha-m-1}e^{-i\nu t}$ converges absolutely, and so by (1.2) and the foot-note $*$),

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*) If in (1.2) x is a complex variable we consider of course the principal value of $(1-x)^{-\gamma-1}$ only, and this identity then holds also when $|x|=1$ provided that $\gamma\leqq-1$.