28. *On Minimal Slit Domains*

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An arbitrary plane domain of finite connectivity is mapped conformally onto the whole plane with the slits parallel to the real axis, and the mapping is uniquely determined except some suitable linear transformations. But when the domain is of infinite connectivity, the uniqueness does not hold. Koebe introduced the concept *minimalen Schlitzbereich* as the normal domain, the mapping onto which is unique (cf. [1, 2]). This mapping function is obtained as the solution of an extremal problem (cf. [3]). In this paper, we deal with the existence of the extremal mapping function onto the circular ring with radial or circular slits as normal domain and some properties of this normal domain, *minimal ring*.

1. Conformal mapping onto a circular ring with the radial slits

Let Ω be a plane domain bounded by $N+2$ analytic Jordan curves $\gamma_1, \gamma_2, \delta_1, \cdots, \delta_N$. We define the class $\mathfrak{H}$ of function h on Ω, satisfying the following conditions:

1) h is harmonic in Ω,
2) there is a negative constant $k(h)$ depending on h, and $h=k(h)$ on γ_2, and $h=0$ on γ_1,
3) for the conjugate harmonic function h^* of h,

$$\int_{\gamma_1} dh^*=2\pi, \quad \int_{\gamma_2} dh^*=-2\pi, \quad \int_{\delta_j} dh^*=0 \qquad (j=1,\cdots,N).$$

Let h_0 be the function of $\mathfrak{H}$ such that

$$\frac{\partial h_0}{\partial n}=0 \quad \text{on } \delta_j \qquad (j=1,\cdots,N).$$

Then, the function $f(z)=e^{h_0+ih_0^*}$ maps Ω onto a circular ring $r_0<|w|<1$ $(r_0=k(h_0))$ with N radial slits in the w-plane.

Lemma 1. *The function h_0 minimizes the functional*

$$2\pi k(h)+\int_{\delta} h\, dh^* \qquad (\delta=\bigcup_j \delta_j)$$

in $\mathfrak{H}$ and the minimizing function is unique.

Proof.
$$D(h-h_0)=\int_{\partial\Omega}(h-h_0)\, d(h-h_0)^*$$
$$=\int_{\delta} h\, dh^*+\int_{\delta}(h\, dh_0^*-h_0\, dh^*)$$