

37. Note on the Cluster Sets of the Meromorphic Functions

By Chuji TANAKA

Mathematical Institute, Waseda University, Tokyo

(Comm. by S. SUTUNA, M.J.A., April 13, 1959)

Let $w=f(z)$ be uniform and meromorphic in the arbitrary connected domain D , whose boundary is C and z_0 be the non-isolated boundary point on C . We denote the part of D and C in $|z-z_0|<r$ by D_r and C_r respectively. As usual, we define the cluster sets at z_0 as follows:

Definition 1. The interior cluster set $S_{z_0}^{(D)}$ is defined by $\bigcap_{r>0} \overline{\mathfrak{D}_r}$, where \mathfrak{D}_r is the set of values taken by $f(z)$ in D_r .

The intersection $\bigcap_{r>0} \mathfrak{D}_r$ is called the range of values $R_{z_0}^{(D)}$. The boundary cluster set $S_{z_0}^{(C)}$ is the intersection $\bigcap_{r>0} M_r$, where M_r is the closure of the union $\bigcup_{z_0 \neq z \in C_r} S_z^{(D)}$.

Definition 2. We denote by $O_{z_0}^{(D)}$ the set of inner points of $S_{z_0}^{(D)}$.

K. Noshiro has proved the

Theorem [1, p. 84]. The set $R_{z_0}^{(D)}$ is everywhere dense in the open set $G=S_{z_0}^{(D)}-S_{z_0}^{(C)}$, provided that G is not empty.

In this note, we shall prove the following extension of K. Noshiro's theorem, i.e. the

Theorem. If $O_{z_0}^{(D)}$ is not empty, then $R_{z_0}^{(D)}$ contains the G_δ set $\bigcap_m O_m$, where O_m is the open set everywhere dense with respect to $O_{z_0}^{(D)}$.

As its corollary we get the

Corollary. If $R_{z_0}^{(D)}$ is empty, then $S_{z_0}^{(D)}$ has no inner point.

The argument of its proof is due to F. Bagemihl [2].

Proof. By a rational disk we mean an open disk whose radius is a rational number and whose center is a complex number with its real and imaginary parts both rational. The set of rational disks contained in $O_{z_0}^{(D)}$ is enumerable. Let its elements be denoted by $D_1, D_2, \dots, D_n, \dots$. According to the definition of $S_{z_0}^{(D)}$, for every natural number n , there exists a point $d_n^{(m)} \in D_n$ and a point $z_n^{(m)} \in D$ such that $f(z_n^{(m)})=d_n^{(m)}$, $0 < |z_n^{(m)} - z_0| < 1/2m$. Hence there exists an open disk $K_n^m \subset D_n$ with $d_n^{(m)} \in K_n^m$, such that every value in K_n^m is assumed by $f(z)$ in $D \cap C_n^m$, where C_n^m is the disk defined by $|z - z_n^{(m)}| < 1/2m$.

Put $O_m = \bigcup_n K_n^m$. It is obvious that O_m is open. O_m is everywhere dense in $O_{z_0}^{(D)}$, because if N is any neighbourhood contained in $O_{z_0}^{(D)}$, then there exists k such that $D_k \subset N$, and since $K_k^m \subset D_k \subset N$, $O_m \cap N$ is not empty.