33. On the Extensions of Finite Factors. I

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In the previous paper [3], we have introduced the concept of crossed product of von Neumann algebras and using it we have generalized the construction of finite factors given by F. J. Murray and J. von Neumann [2]. However the definition given there is rather restricted comparing with the usual concept given for Notherian rings since it corresponds only for cases with special factor sets. In this note we modify the definition of crossed product of von Neumann algebras to fill up this gap and show some properties of this generalized product.

We shall show in the below that the crossed product of a finite factor with respect to a *normalized* factor set of unitary operators can be defined as usual and that the product is related deeply with the extension of a group described in [1]. Also we shall show that the product satisfies the usual elementary properties of the crossed product and that our new product gives a way of construction of the crossed product basing on an intermediate subfactor.

1. Let $\tilde{\mathfrak{A}}$ be a discrete group and K be a normal subgroup of $\tilde{\mathfrak{A}}$. Put $\mathfrak{A}$ the quotient group $\tilde{\mathfrak{A}}/K$. Now we assume that a group G is isomorphic to a subgroup of $\mathfrak{A}$. (As there is no fear for confusion, we identify G with the subgroup in the following.) For each $\alpha \in G$, we define a selection $\bar{\alpha} \in \alpha$. Throughout the note, the selection $\alpha \to \bar{\alpha}$ will be fixed unless the contrary is stated explicitly. Since K is normal, $k^{\alpha} = \bar{\alpha}^{-1} k \bar{\alpha} \in K$ for every $k \in K$ and for every $\alpha \in G$, and so α determines an automorphism of the group K. (Every element m of K induces an inner automorphism of K such that $k^m = m^{-1} k m$.) By the choice of $\bar{\alpha}$ and $\bar{\beta}$, there is an element $m_{\alpha,\beta} \in K$ satisfying the relation

$$\bar{\alpha} \cdot \bar{\beta} = \overline{\alpha\beta} \cdot m_{\alpha,\beta}.$$

Considering α and β as automorphisms of K, the above relation gives

$$(1) \qquad (k^{\alpha})^{\beta} = (k^{\alpha\beta})^{m_{\alpha,\beta}} \qquad \text{for } k \in K.$$

From the definition of the automorphism γ of K, $m_{\alpha,\beta}\bar{\gamma} = \bar{\gamma} m_{\alpha,\beta}^{\gamma}$. By this equality and the associativity of group operation for $\tilde{\mathfrak{A}}$, the family $\{m_{\alpha,\beta} \mid \alpha, \beta \in G\}$ of elements in K satisfies the following equality:

$$(2) \qquad m_{\alpha,\beta\gamma} m_{\beta,\gamma} = m_{\alpha\beta,\gamma} m_{\alpha,\beta}^{\gamma}$$

for every α, β and γ in G. Such a family $\{m_{\alpha,\beta} \mid \alpha, \beta \in G\}$ satisfying (2)