59. On Ring Homomorphisms of a Ring of Continuous Functions

By Takesi ISIWATA

Tokyo Gakugei University, Tokyo

(Comm. by K. KUNUGI, M.J.A., June 12, 1959)

Let γ be a linear subring of $C(X)$ and $H(\gamma)$ the totality of non-trivial ring homomorphisms[1] on γ and $H_0(\gamma)=H(\gamma)\smile\{\varphi_0\}$ where φ_0 denotes the trivial ring homomorphism, i.e. $\varphi_0=0$ on γ. We shall say that γ has the property (H) if the following property holds:

(H). *any* $\varphi\in H(\gamma)$ *is a point ring homomorphism* φ_x.[1] If $H(\gamma)$ is replaced by $H_0(\gamma)$, then we shall call that γ has the property (H_0). In case the property (H) or (H_0) holds respectively, if $\varphi_x\neq\varphi_y$ for $x\neq y$, then we say that γ has the property (H^*) or (H_0^*) respectively. Ishii [1] and Mrókwa [2] have obtained necessary and sufficient conditions that a subring γ containing constant functions has the property (H^*) or (H) respectively, under some conditions on γ. We denote by (h) one of the properties (H^*), (H), (H_0^*) and (H_0). In this paper we shall give a necessary and sufficient condition that $C(X)$ has a linear subring on which the property (h) is satisfied.[2] Moreover, we shall generalize Ishii's and Mrókwa's results, and give a weaker condition for which γ has the property (H).

1. Suppose that γ is a linear subring which has the property (h). Let us put

$$\hat{x}=\{y;\ f(x)=f(y)\text{ for all } f\in\gamma\}.$$

$\hat{x}$ is a closed subset of X because $\hat{x}=\bigcap_{f\in\gamma}f^{-1}(\alpha)$ where $f(x)=\alpha$. Then X is divided into a family $\hat{X}=\{\hat{x};\ x\in X\}$ of disjoint closed subsets of X. We shall define uniform neighborhoods of $\hat{x}$ by

$$W(f_1,\cdots,f_n;\varepsilon)(\hat{x})=\{\hat{y};\ |f_i(x)-f_i(y)|<\varepsilon,\ i=1,2,\cdots,n\}$$

where $f_i\in\gamma$ and $\varepsilon>0$. Then X becomes a uniform space with the uniform basis $\{W(f_1,\cdots,f_n;\varepsilon);f_i\in\gamma,\ \varepsilon>0\}$. In the following we denote by $Y=X/\gamma$ such a uniform space.

Let η be a mapping of X into Y defined by $\eta(x)=\hat{x}$ and η^* be a mapping of $C(Y)$ into $C(X)$ defined by $(\eta^*f)(x)=f(\eta(x))$ where $f\in C(Y)$ and $x\in X$. Then η is continuous and $\eta(x)=\eta(y)$ for any $y\in\hat{x}$ implies

1) A space X considered here is always a completely regular T_1-space, and other terminologies used here, for instance, $C(X)$, $B(X)$, ring homomorphisms and local Q-completeness, are the same as in [6, 7].

2) If we mean by $H(\gamma)$ the totality of non-trivial ring homomorphisms of γ *into* R, then we can replace by a subring a linear subring in Theorem 1 and Corollaries 1-5.