58. On Locally Q-complete Spaces. II

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1. In this paper we shall consider the problems characterizing a given space X by a ring of continuous functions defined on X.[1] Shirota [1] has proved that if X is a Q-space, then $C(X)$ characterizes X, that is, the ring isomorphism of $C(X)$ onto $C(Y)$ implies that X is homeomorphic to Y for any Q-spaces X and Y. In general, it is easy to see that $C(X)$ or $B(X)$ does not characterize X. But under some conditions on a ring isomorphism this problem is solved in the affirmative [2, 6]. On the other hand, Shanks [3] and Ishii [4] have proved that if X is locally compact, then $C_k(X)$[2] characterizes X.

In this paper, we shall generalize Shanks' theorem and it will be shown that for any locally Q-complete space X which is not compact, there is a subring of $C(X)$ on which any non-trivial ring homomorphism[3] is a point ring homomorphism. Moreover we shall prove that such a subring characterizes X.

2. Extension of functions

Let $f\in C(X)$ and $\tilde{f}$ be a continuous extension over βX of f (if it exists, i.e. f is bounded). If f can be continuously extended over a point $p\in\beta X-X$, f has a finite value at the point p. If f is not continuously extended over the point p, then for any $m>0$, $f_m=(f\wedge m)\vee(-m)$[4] has a continuous extension $\tilde{f}_m$ because f_m is bounded. It is easily seen that $\tilde{f}_m(p)=m$. Therefore there exists a neighborhood (in X)[5] of the point p on which $f>n$ for a given integer $n>0$. Let

1) A space X considered here is always a completely regular T_1-space, and other terminologies used here, for instance $C(X)$, are the same as in [7].

2) $C_k(X)$ is a ring consisting of all continuous functions which have compact supports.

3) A non-trivial *ring homomorphism* of a subring C_1 of $C(X)$ means a ring homomorphism of C_1 *onto* R where R is a ring of all real numbers. But a ring homomorphism is not necessarily *linear*, for C_1 does not necessarily contain constant functions. A point ring homomorphism φ is defined by $\varphi(f)=f(p)$ for all $f\in C_1$ where p is a fixed point in X. In this case φ is completely determined by the point p, and hence we shall write $\varphi=\varphi_p$. A ring homomorphism φ is called to be *trivial* if $\varphi(f)=0$ for all $f\in C_1$.

4) For any constant m, where no confusion will arise, we use the same letter m for a function which takes a constant value m on X. The symbols "$\vee$" and "$\wedge$" are used in the following sense:

$$(f\vee g)(x)=\max(f(x),g(x)) \quad\text{and}\quad (f\wedge g)(x)=\min(f(x),g(x)).$$

5) *A neighborhood* (*in* X) *of* x^* means a set U such that $U=X\frown V$ where V is a neighborhood of x^* in βX.