81. On the Singularity of a Positive Linear Functional on Operator Algebra

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In the previous paper [3], we have introduced the notion of a singular linear functional on W^*-algebra $\boldsymbol{M}$ as follows: a positive linear functional φ on $\boldsymbol{M}$ is called *singular* if there exists no non-zero σ-weakly continuous positive linear functional ψ such as $\psi \leqq \varphi$. This notion is corresponding to the one of *purely finite additive measure* in the abelian case of Yosida-Hewitt [5]. And we have proved the decomposition theorem of positive linear functional on $\boldsymbol{M}$, whose another proof was given by Nakamura in [2], as follows: *Any positive linear functional φ on $\boldsymbol{M}$ is uniquely decomposed into the sum of σ-weakly continuous positive linear functional φ_1 and singular one φ_2. And if φ is singular, then φ is so on $p\boldsymbol{M}p$ for every non-zero projection p of $\boldsymbol{M}$.* Moreover, suppose $\boldsymbol{M}_*$ is the space of all σ-weakly continuous linear functionals on $\boldsymbol{M}$ and $\boldsymbol{M}_*^{\perp}$ the space of all linear combinations of singular positive linear functionals, we have proved the following decomposition of the conjugate space $\boldsymbol{M}^*$ of $\boldsymbol{M}$: $\boldsymbol{M}^* = \boldsymbol{M}_* \oplus_{l^1} \boldsymbol{M}_*^{\perp}$ *where $\oplus_{l^1}$ means the l^1-direct sum of its summands.* This decomposition of the conjugate space implies that of a uniformly continuous mapping which proved by Tomiyama [4] as follows: *Let π be a uniformly continuous linear mapping from $\boldsymbol{M}$ into another W^*-algebra $\boldsymbol{N}$, then there exist unique two linear mappings π_1 and π_2 of $\boldsymbol{M}$ into $\boldsymbol{N}$ such that $\pi = \pi_1 + \pi_2$, π_1 is σ-weakly continuous and ${}^t\pi_2(\boldsymbol{N}_*) \subset \boldsymbol{M}_*^{\perp}$ where ${}^t\pi_2$ means the transpose of π_2. And according to π being a homomorphism, positive or *-preserving, π_1 and π_2 are homomorphisms, positive or *-preserving respectively.* Hence a linear functional φ on M and a linear mapping π from $\boldsymbol{M}$ into another W^*-algebra $\boldsymbol{N}$ are called *singular* if $\varphi \in \boldsymbol{M}_*^{\perp}$ and ${}^t\pi(\boldsymbol{N}_*) \subset \boldsymbol{M}_*^{\perp}$, respectively.

This note is devoted to give a characterization of the singularity of a positive linear functional on $\boldsymbol{M}$ and a short alternative proof of Theorem 6 in [3].

Theorem 1. *Let $\boldsymbol{M}$ be a W^*-algebra and φ a positive linear functional on $\boldsymbol{M}$. Then a necessary and sufficient condition that φ is singular is that for any non-zero projection e, there exists a non-zero projection $f \leqq e$ such as $\langle f, \varphi \rangle = 0$.*

Proof. Suppose φ is not singular, the σ-weakly continuous part φ_1 of φ is not zero by Theorem 3 in [3]. Let e be the carrier pro-