80. *Normal Operators in Hilbert Spaces and Their Applications*

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In the present note, we wish to outline the following three problems for a compact or non-compact normal operator N in Hilbert space $\mathfrak{H}$ which is complete, separable, and infinite-dimensional:

1° the problem of finding characteristic elements and their corresponding characteristic values of N;

2° the problem of finding the multiplicities of characteristic values of N;

3° the problem of finding analytical properties of some normal operators associated with N.

The results which we shall give can be applied to linear non-homogeneous integral equations with normal kernels, but we will only give a few examples here.

The details will be shortly published in Memoirs of the Faculty of Education of Kumamoto University.

As a first step, we can easily establish the following lemmas:

Lemma 1. If N is a compact normal operator in $\mathfrak{H}$, then

(A) any non-null complex number different from all characteristic values of N belongs to the resolvent set;

(B) supposing that $\{\lambda_\nu\}_{\nu=1,2,\cdots}$ is the sequence of all characteristic values of N, arranged in an order such that $|\lambda_1|\geqq|\lambda_2|\geqq\cdots$, and denoting by E_ν the characteristic projector of N corresponding to λ_ν, $N=\sum_\nu \lambda_\nu E_\nu$ where the right-hand member converges uniformly to N in the case that $\{\lambda_\nu\}$ is an infinite sequence.

Lemma 2. Let N be a compact normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}$ and $\{E_\nu\}$ be the same symbols as those used in Lemma 1 respectively; and let $f^{(k)}$ be an arbitrary element of $\mathfrak{H}$ such that $E_\nu f^{(k)}=0$ for $\nu=1,2,\cdots, k-1$ and $E_k f^{(k)}\neq 0$. Then

$$|\lambda_k|=\lim_{n\to\infty}\| N^n f^{(k)}\|^{\frac{1}{n}}.$$

Lemma 2 here can be derived by a utilization of the expansion $N=\sum_\nu \lambda_\nu E_\nu$ in Lemma 1.

Put

$$g^{(k)}\equiv\frac{\sum_{\nu=k}^{p} E_\nu f^{(k)}}{\left\|\sum_{\nu=k}^{p} E_\nu f^{(k)}\right\|}\quad(1\leqq k\leqq p<\infty),\quad f_n\equiv(N^*N)^n f^{(k)},$$