79. *On Fatou's Theorem*

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1. As one of the classical theorems in the theory of functions, the following Fatou's theorem is well known:

"If $f(z)$ is regular and bounded in the unit circle, then at almost all points of the unit circle the boundary values of $f(z)$ exist".

It seems to me that the proof, due to Carathéodory, is based on the fact that the boundary value is a differential coefficient of a function which satisfies "Lipschitz condition". In this paper we shall get into an argument so that the boundary value should be a differential coefficient of a function VBG_*.[1)]

2. After this, we shall consider the function $f(z)$, one-valued regular in the unit circle: $|z|<1$. First of all, we pose the following condition (A):

(A) on the unit circle $C: |z|=1$, there exists a closed set N such that

(i) mes. $N=0$[2)]

(ii) $\sup\limits_{0\leq r<1}|f(re^{i\theta})|<\infty$ for $\theta\in C-N$.[3)]

Proposition 1. *Under the condition* (A), *if we set*

$$F(\rho,\theta)=\int_{P_0}^{\theta}f(\rho e^{i\varphi})d\varphi,\quad \theta_0\notin N,\quad 0\leq\rho<1,$$

then for every $\theta\in C-N$ *there exists the limit*:

$$\lim_{\rho\to 1}F(\rho,\theta).$$

Proof. As is easily seen, we can set $f(0)=0$, and suppose that there exists a sequence of sets $\{E_n\}$, such that

(1°) $\sum\limits_{n=1}^{\infty}E_n=C-N$,

(2°) if $\theta\in E_n$ then $\sup\limits_{0\leq\rho<1}\left|\dfrac{f(\rho e^{i\theta})}{\rho e^{i\theta}}\right|\leq n_0+n$,

(3°) $\theta=0\in E_1$ $(\notin N)$.

We shall set $A=\rho$, $B=\rho+\Delta\rho$, $C=\rho e^{i\theta}$, $D=(\rho+\Delta\rho)e^{i\theta}$, $(0\leq\rho<\rho+\Delta\rho<1)$ then

$$F(\rho+\Delta\rho,\theta)-F(\rho,\theta)=\int_0^{\theta}f\{(\rho+\Delta\rho)e^{i\varphi}\}d\varphi-\int_0^{\theta}f(\rho e^{i\varphi})d\varphi$$

$$=\int_B^D\frac{f(z)}{iz}dz-\int_A^C\frac{f(z)}{iz}dz,$$

1) Cf. S. Saks: Theory of the Integral.

2) mes. N means the measure of the set N.

3) $C-N=\{\theta:\ \theta\in C,\ \theta\notin N\}$.