126. On Equivalence of Modular Function Spaces

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1959)

Let Ω be an abstract space and μ be a totally additive measure defined on a totally additive set class $\mathfrak{B}$ of subsets of Ω satisfying $\bigcup_{\mu(E)<\infty} E=\Omega$.

Let $\Phi(\xi, \omega)$ ($\xi \geqq 0$, $\omega \in \Omega$) be a function satisfying the following conditions:

1) $0 \leqq \Phi(\xi, \omega) \leqq \infty$ for all $\xi \geqq 0$, $\omega \in \Omega$;
2) $\Phi(\xi, \omega)$ is a measurable function on Ω for all $\xi \geqq 0$;
3) $\Phi(\xi, \omega)$ is a non-decreasing convex functions of $\xi \geqq 0$ for all $\omega \in \Omega$;
4) $\Phi(0, \omega)=0$ for all $\omega \in \Omega$;
5) $\Phi(\alpha-0, \omega)=\Phi(\alpha, \omega)$ for all $\omega \in \Omega$;
6) $\Phi(\xi, \omega) \to \infty$ as $\xi \to \infty$ for all $\omega \in \Omega$;
7) for any $\omega \in \Omega$, there exists $\alpha_\omega > 0$ such that $\Phi(\alpha_\omega, \omega) < \infty$.

For any measurable function $x(\omega)$ ($\omega \in \Omega$), $\Phi(|x(\omega)|, \omega)$ is also measurable. We shall denote by $L_\Phi(\Omega)$ the class of all measurable functions $x(\omega)$ ($\omega \in \Omega$) such that, for some $\alpha=\alpha_x>0$,

$$\int_\Omega \Phi(\alpha |x(\omega)|, \omega) d\mu(\omega) < \infty.$$ [1]

We write $x \geqq y$ ($x, y \in L_\Phi$), if $x(\omega) \geqq y(\omega)$ for a.e.[2] on Ω, then L_Φ is a universally continuous semi-ordered linear space.

If we define a functional

$$m_\Phi(x)=\int_\Omega \Phi(|x(\omega)|, \omega) d\mu,$$

m_Φ satisfies all the modular conditions and furthermore m_Φ is monotone complete. Such a space L_Φ with m_Φ is said to be *a modular function space*.[3]

If $\overline{\Phi}(\eta, \omega)$ ($\eta \geqq 0$, $\omega \in \Omega$) is, for every fixed $\omega \in \Omega$, the complementary function of Φ in the sense of H. W. Young, $\overline{\Phi}$ satisfies all the corresponding properties from 1) to 7) on Φ, and so, we have also a

1) For the integration, refer, for instance, H. Nakano [4].

2) Here "a.e. (almost everywhere)" means always that "except on some $A \in \mathfrak{B}$ which $\mu(E \cap A)=0$ for all $\mu(E)<\infty$".

3) Modulared function spaces were defined and discussed in H. Nakano [2, Appendices I, II]. For all other definitions and notations used in this note, see the same book, too.