134. *Representation of Some Topological Algebras. III*

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7. On the condition (ii). It is easy to see that a semi-simple algebra satisfies the condition $(*)$[1] but not the condition (ii)[1] in general. Let E be an algebra, and let $u\in E$; then we denote by $(u)_r$ the right ideal generated by u, that is, the set of all elements $\lambda u+ux$, where λ runs over the scalar field and x over the whole E; we write $(u)_l$ the left ideal generated by u.

LEMMA 1. *For a semi-simple algebra E, each one of the following conditions is equivalent to the condition* (ii):

(1) *For any two non-zero elements $u, v\in E$, we have $uE\frown Ev \neq \{0\}$.*

(2) *For any two non-zero elements $u, v\in E$, we have $(u)_r\frown(v)_l \neq \{0\}$.*

Proof. It is clear that the condition (ii) implies (1) and (1) implies (2). To prove the implication (2)$\rightarrow$(ii), let us suppose that an algebra E satisfies the condition (2) and not (ii). Then there exist two non-zero elements $u, v\in E$ such that $uxv=0$ for every $x\in E$. Since E is semi-simple, we can find an element $a\in E$ with $ua\neq 0$, and so by (2), there exists a non-zero element $w=\alpha ua+uab=\beta v+cv\in(ua)_r\frown(v)_l$, where α, β are two numbers and $b, c\in E$. Now, if $w^2=0$, then for any number λ and any $x\in E$, we have

$$\lambda w+xw-\lambda w-xw+\lambda^2w^2+\lambda wxw+\lambda xw^2+xwxw=0,$$

since $wxw=(\alpha ua+uab)x(\beta v+cv)=0$; it follows that w belongs to the radical of E, and so $w=0$, which is a contradiction. Thus $w^2=\alpha\beta uav+\alpha uacv+\beta uabv+uabcv\neq 0$. But this is absurd since $uEv=\{0\}$.

LEMMA 2. *For an algebra E with a minimal left ideal L such that $L^2\neq\{0\}$, each one of the following conditions is equivalent to the condition* (ii):

(1) *For any non-zero element $u\in E$, we have $uE\frown L\neq\{0\}$.*

(2) *For any non-zero element $u\in E$, we have $uL\neq\{0\}$.*

Proof. Since $L^2\neq\{0\}$, we can find an idempotent $p\in E$ such that $L=Ep$. The implication (ii)$\rightarrow$(1) is obvious, because $uEp\subseteq uE\frown L$. If there exists a non-zero element $ux\in uE\frown Ep$, then we have $ux=ap$ for some $a\in E$, and hence $0\neq ap=uxp\in uL$, proving the implication (1)$\rightarrow$(2). Now suppose that the condition (2) is satisfied, and let u, v

1) Cf. S. Kasahara: Representation of some topological algebras. I, II, Proc. Japan Acad., **34**, 355–360 (1958); **35**, 89–94 (1959).