10. On a Problem of Royden on Quasiconformal Equivalence of Riemann Surfaces

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1. **Definitions and problem.** We denote by $HBD(R)$ the totality of complex-valued bounded harmonic functions on a Riemann surface R with finite Dirichlet integrals. We use the following convention. If R is of null boundary, then the complex number field C is considered not to be contained in $HBD(R)$, that is, the constant function is not HBD-function and hence $HBD(R)$ is empty. On the other hand, if R is of positive boundary, then C is considered to be contained in $HBD(R)$.

Now consider the set $A(R)$ of all bounded and continuously differentiable functions on R with finite Dirichlet integrals. Then there exists a compact Hausdorff space $\widetilde{R}$ containing R as its open dense subset and any function in $A(R)$ is continuously extended to $\widetilde{R}$. Such a space $\widetilde{R}$ is unique up to a homeomorphism fixing R. The set $\partial R=\widetilde{R}-R$ is called the *ideal boundary* of R.

Let $\{R_n\}_{n=0}^{\infty}$ be an exhaustion of R with R_0=empty set. For each n, consider the family $\{F^{(n)}\}$ of closed subsets $F^{(n)}$ of $\widetilde{R}-R_n$ such that any real-valued continuous function on $\widetilde{R}-R_n$, which belongs to $HBD(R-\overline{R}_n)$, takes its maximum and minimum on $F^{(n)}$. The set

$$\bigcap_{n=0}^{\infty}\bigcap_{\{F^{(n)}\}} F^{(n)}$$

is empty or the compact subset of ∂R. We denote this set by $\partial_1 R$. Denote by $A_1(R)$ the totality of functions in $A(R)$ which vanish on $\partial_1 R$. Then any function f in $A(R)$ is decomposed into two parts u in $HBD(R)$ and $f-u$ in $A_1(R)$. This decomposition is unique and so we denote u by πf. Then it holds that

$$D[\pi f, f-\pi f]=\iint_R d(\pi f)\wedge {}^*\overline{d(f-\pi f)}=0.$$

Consider the following algebraic operations in $HBD(R)$: for arbitrary two functions u and v in $HBD(R)$ and for any complex number a, we define *addition*, *scalar multiplication* and *multiplication* by the following

$$(u+v)(p)=u(p)+v(p);$$
$$(au)(P)=a(u(p));$$
$$(u\times v)(p)=(\pi(uv))(p),$$