# 8. *On Transformation of Manifolds*

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Let $m>n>r\geq 1$ be integers, suppose $M$ is an $m$-dimensional and $N$ an $n$-dimensional oriented closed polyhedral manifold, let $S$ be the simplicial image of an oriented $r$-sphere situated in $N$, and $f:M\rightarrow N$ a continuous mapping. Then one may suppose that $f^{-1}(S)$ is a finite polyhedron $R$ in $M$ satisfying

$$\dim R=m-n+r.$$

Let $A_1, A_2,\cdots$ be the $(m-n+r)$-simplexes of a simplicial decomposition of $R$, moreover $A$ one of the $A_i$, and $A^*$ an orientation of $A$. The simplexes used here are open and rectilinear. If $a$ is a point in $A$, one can suppose $S$ is smooth in a neighborhood of the point $b=f(a)$. Let $B$ be an $r$-simplex with $b\in B\subset S$. Define $C$ to be an $(n-r)$-simplex in $M$ perpendicular to $A$, and $D$ an $(n-r)$-simplex in $N$ perpendicular with respect to $B$ such that $A\cap C=a$, $B\cap D=b$, $R\cap\bar{C}=a$, and $S\cap\bar{D}=b$. For every point $p\in\partial C$, let $\varphi(p)$ denote the vertical projection of $f(p)$ on $D$ parallel to $B$. Then $\varphi(\partial C)\subset D-b$. For $p\in\partial C$, let $\varphi'(p)$ be the vertical projection of $\varphi(p)$ on $\partial D$ out of $b$. By $C^*$ we denote an orientation of $C$ such that $(A^*, C^*)$ gives the positive orientation of $M$, by $B^*$ the orientation of $B$ induced by $S$, and by $D^*$ an orientation of $D$ such that $(B^*, D^*)$ furnishes the positive orientation of $N$. Let $\beta(A^*)$ be the Brouwer degree of the map $\varphi':\partial B^*\rightarrow\partial D^*$.

Let $a_k$ be an orientation of $A_k$ and $\beta_k$ the number $\beta(a_k)$. Then $\sum\beta_k a_k$ represents a finite $(m-r+r)$-cycle that we will denote by $\sigma_f(S)$ as well. If the continuous $r$-sphere $S'$ is homotopic to $S$ within $N$, then

$$\sigma_f(S)\sim\sigma_f(S').$$

Let $\pi_r(N)$ be the $r$-dimensional Hurewicz group of $N$. Define $h$ to be the homotopy class of $S$, and $\zeta(h)$ to be the homology class of $\sigma_f(S)$. Then the mapping $\zeta:\pi_r(N)\rightarrow H_{m-n+r}(M)$, where $H_i(M)$ means the $i$-dimensional integral Betti group of $M$, is a homomorphism. Of course, the latter is related to known inverse homomorphisms. But for the following it is important to have an exact *geometric realization* of these homomorphisms; a problem to which already Whitney [4] has hinted.

Now suppose $r=2n-m-1\geq 2$, and let $\pi_r^\zeta(N)$ be the kernel of the homomorphism $\zeta$, moreover $h_r^\zeta$ an element of $\pi_r^\zeta(N)$, and $Q$ an oriented continuous sphere of $h_r^\zeta$. One may suppose $f^{-1}(Q)$ is an $(m-n+r)$-polyhedron in $M$. Denote the cycle $\sigma_f(Q)$ by $z$ as well. Evidently,