35. A Certain Type of Vector Field. III

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The objective of the present paper is to prove the theorem announced in section V of the preceding paper [1].

To avoid the trivial repetition of the same technique of proving we shall verify only the fact that the existence of a vector field (12) of the above place is equivalent to that of such a conformal separability as this:

(1) $$ds^2=\sinh^2(cx^n+d)\,ds_0^2+(dx^n)^2$$

where c and d are constants, and ds_0^2 is an $(n-1)$-dimensional metric form independent of x^n.

First let us assume that the metric form is conformally separable in the way of (1). Set $\xi_i=\delta_i^n$, where δ_i^n is the so-called Kronecker's delta. Then we have

$$\xi_{i|j}=c\coth(cx^n+d)\,g_{ij}-c\coth(cx^n+d)\,\xi_i\xi_j.$$

Let $V_i=\tanh(cx^n+d)$ and we get

$$\begin{aligned}V_{i|j}&=\tanh(cx^n+d)\,\xi_{i|j}+c\,\mathrm{sech}^2(cx^n+d)\,\xi_i\delta_j^n\\&=c\,g_{ij}-c\{1-\mathrm{sech}^2(cx^n+d)\}\xi_i\xi_j\\&=c\,g_{ij}-c\tanh^2(cx^n+d)\xi_i\xi_j=c(g_{ij}-V_iV_j).\end{aligned}$$

The converse is as follows. Suppose that V satisfies (12) of [1]. Then we have

(2) $$\tfrac{1}{2}(\|V\|^2)_{|j}=c(1-\|V\|)^2V_j.$$

Taking a canonical coordinate to V, we have

$$V^i=\|V\|^2\delta_n^i \quad\text{and}\quad g_{nn}=\frac{1}{\|V\|^2}.$$

From (2) we get

$$\tfrac{1}{2}(\|V\|^2)_{|n}=c(1-\|V\|^2).$$

Consequently

$$\frac{\|V\|_{|n}}{1-\|V\|^2}=c\sqrt{g_{nn}}.$$

Hence we find

(3) $$\|V\|=\tanh(cs+d),$$

where s is the arc length of the tangent curve. It is easily seen that d is a constant. From (10) of [1] we have

$$\begin{aligned}H(x)&=\exp 2\int\frac{c}{\tanh(cs+d)}\sqrt{g_{nn}}\,dx^n\\&=\exp 2c\int\frac{ds}{\tanh(cs+d)}=\sinh^2(cs+d).\end{aligned}$$