62. *Harmonic Analysis on the Group of Linear Transformations of the Straight Line*

By Akihiro KOHARI

(Comm. by K. KUNUGI, M.J.A., May 13, 1961)

Introduction. Let G be the group of linear transformations of the straight line: $\xi \to g(\xi)=a\xi+b, (a>0, -\infty<b<\infty, -\infty<\xi<\infty)$. The irreducible unitary representations of G are described by I. M. Gelfand and M. A. Naimark (see the reference [1]). In the present paper, we shall investigate harmonic analysis on this group in more details and consider (1) spherical functions, (2) a generalization of the Laplace operator in the homogeneous space, (3) canonical bases of the representation spaces, (4) canonical Lie operators, and (5) an expansion formula by means of irreducible representations. We give the exact definitions of these notions in the following.

Let N be the subgroup $\{b \in G : b(\xi)=\xi+b\}$, and $\mathfrak{H}$ the Hilbert space of all complex-valued square integrable functions on the character group of N with respect to the usual Lebesgue measure. Then, for $f \in \mathfrak{H}$, the operators

$$U_g f(x)=\sqrt{a}\, e^{ibx} f(ax) \tag{0.1}$$

define a unitary representation of G. It is decomposed into the direct sum of two irreducible representations $\mathfrak{D}_\pm : \{U_g; \mathfrak{H}_\pm\}$, where the elements of $\mathfrak{H}_\pm$ are functions vanishing almost everywhere on the left or the right half line respectively. Any irreducible unitary representation is equivalent to either $\mathfrak{D}_+$, $\mathfrak{D}_-$ or a one dimensional representation of the subgroup $H=\{h: h(\xi)=a\xi\}$, [1]. We shall not consider the latter representations, for they have no significance to our problems, and except in 2, we shall consider only $\mathfrak{D}_+$ because all arguments are quite similar for $\mathfrak{D}_-$.

1. **Spherical functions.** For a moment let G be an arbitrary Lie group and H be a closed subgroup of G, and let $\mathfrak{D}:\{U_g; \mathfrak{H}\}$ be an irreducible unitary representation of G. If there exists a topological vector space $\mathfrak{F}$ which is dense in $\mathfrak{H}$ and a non-zero linear functional θ over $\mathfrak{F}$ such that

$$U_h^* \theta=\theta \quad \text{for all} \quad h \in H, \tag{1.1}$$

where U_h^* is the adjoint operator of U_h, then for $f \in \mathfrak{F}$, the function $\tilde{f}(g)=(f, U_g^* \theta)$ is constant on every right coset of G with respect to H. We shall call these functions *the spherical functions belonging to* $\mathfrak{D}$ (cf. [2]). In the present case, let $\mathfrak{F}_+$ be the space of all C^∞ functions whose carriers are compact and in the right half line, and