59. *Heisenberg's Commutation Relation and the Plancherel Theorem*

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1. Let G and X be a locally compact abelian group and its character group, with the Haar measures dg and $d\chi$, respectively. For a Borel subset S of G

(1) $$E(S)f(g)=C_S(g)f(g),$$

where $C_S(g)$ is the characteristic function of S, defines a spectral measure dE acting on $L^2(G)$. It is easy to see that dE satisfies

(2) $$U(g)E(S)=E(gS)U(g),$$

for the regular representation $U(g)(f(\cdot)\rightarrow f(g^{-1}\cdot))$ of G on $L^2(G)$. Using dE, one can define

(3) $$V(\chi)=\int\overline{\chi(g)}dE(g),$$

for each character $\chi\in X$, where the integration ranges over G. It is not hard to see that $V(\chi)$ is a strongly continuous unitary representation of X. The pair $U(g)$ and $V(\chi)$ satisfies the so-called *Heisenberg's commutation relation*:

(4) $$U(g)V(\chi)=\chi(g)V(\chi)U(g).$$

The representations of a pair of unitary groups satisfying (4) are discussed initially by M. H. Stone [4] and J. von Neumann [3] for n-parameter cases. Their Theorem is generalized to locally compact abelian separable groups by G. W. Mackey [2] and improved away the separability by L. H. Loomis [1], which is stated as the following way: *Let $U'(g)$ and $V'(\chi)$ be strongly continuous unitary representations of G and X on a Hilbert space, respectively, satisfying Heisenberg's commutation relation* (4), *then, according to the pair $U'(g)$ and $V'(\chi)$ being irreducible or not, that pair is unitarily equivalent to the pair of the representations $U(g)$ and $V(\chi)$ or to direct sum of their replicas.* This theorem will be referred as Mackey-Loomis' Theorem.

The purpose of the present note is to show that Heisenberg's commutation relation (4), i. e. Mackey-Loomis' Theorem, implies the Plancherel Theorem. Since the proof of Mackey-Loomis does not assume the duality theorem, our task may be observed with some interests.

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