108. On Information in Operator Algebras

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1. In the present paper, we shall introduce a non-commutative information in an operator algebra. This may be useful for the theory of entropy in quantum statistics (cf. Nakamura-Umegaki [5]).

Let A be a von Neumann algebra with a faithful normal trace τ, and $L^p=L^p(A)=L^p(A,\tau)$ $(p\geqq 1)$ be Banach space consisting of all measurable operators a with the finite integral $\tau(|a|^p)<+\infty$, where the norm is defined by $\|a\|_p=(\tau(|a|^p))^{1/p}$ (cf. Dixmier [1], Segal [6]).

Let S be a set of all normal states $\sigma, \rho, \cdots$ of A. For any $\sigma\in S$, there exists uniquely an operator $d(\sigma)\in L^1$ such that

$$\sigma(a)=\tau(d(\sigma)a) \quad \text{for every } a\in A.$$

The operator $d(\sigma)$ is so-called Radon-Nikodym derivative (of σ with respect to τ), this is due to Dye [2].

For the real valued function $h(\lambda)$ $(\lambda\geqq 0)$ such that

$$(1) \qquad h(\lambda)=-\lambda\log\lambda \ (\lambda>0), \ =0 \ (\lambda=0),$$

an operator function $h(a)$ is defined by

$$h(a)=\int_0^\infty h(\lambda)dE_\lambda$$

for $a\in L^1$ with the spectral resolution $a=\int_0^\infty \lambda dE_\lambda$. Denote

$$(2) \qquad H(a)=\tau(h(a))$$

and it is called *entropy* of the operator a (cf. Nakamura-Umegaki [4]). For any $\sigma\in S$, the entropy $H(d(\sigma))$ of $d(\sigma)$ is denoted by $H(\sigma)$ and it is called the *entropy* of the state σ (cf. Segal [7]).

Segal [7] has proved that the function $H(\sigma)$ over S is concave, and Nakamura-Umegaki [4] has generalized it such as the operator function $h(a)$ over $\{a\in A;\ a\geqq 0\}$ is concave, i.e.

$$(3) \qquad h(\alpha a+\beta b)\geqq \alpha h(a)+\beta h(b)$$

for every $a, b\in A$, $a, b\geqq 0$ and $\alpha, \beta\geqq 0$, $\alpha+\beta=1$. The inequality (3) is extended to the operators $a, b\in L^1$, $a, b\geqq 0$. The entropy $H(a)$ of $a\geqq 0$ is uniquely determined as $-\infty\leqq H(a)\leqq 1$ by a and the trace τ. While, the entropy $H(\sigma)$ of $\sigma\in S$ is determined only by σ and independent from the choice of τ.

2. In the theory of information, various methods have been introduced and discussed by several authors. In the present case we shall introduce into the von Neumann algebra A the amount of information of Kullback-Leibler [3].