145. Approximation of Solutions of Homogeneous Differential Equation. I

By Hideo YAMAGATA

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1961)

§ 1. Hereafter we will use these notations:

$$P(D)\,U=0, \tag{1}$$

where $D=(D_1,\cdots,D_n)=\left(\frac{1}{i}\partial_{x_1},\cdots,\frac{1}{i}\partial_{x_n}\right)$, a homogeneous differential equation with constant coefficient,

$C^n=C\times C\times\cdots\times C$, a direct product of N complex planes,

$R^n=R\times R\times\cdots\times R$, a direct product of N real ax'es,

(D), the set of functions $U\in C^\infty$ with compact carrier,

(D'), the set of functionals on (D) in the sense of L. Schwartz.

We begin with a definition:

Definition 1. A solution of (1) in R^n is called an exponential solution if it can be written in the form

$$U(X)=f(X)\,e^{i\langle X\,\zeta\rangle} \tag{2}$$

where $\zeta\in C^n$ and $f(X)$ is a polynomial.

Approximation of solutions of (1) by the set of exponential solutions has been discussed by Lars Hörmander and B. Malgrange, etc. The following theorem is evident from Hörmander's result.

Theorem 1. The closed linear bull in (D') of the exponential solutions of (1) consists of all solutions of (1) in (D').

In the following, we will give an approximation of solutions of (1) by the smaller set of solutions which consist of complex linear combinations of exponential solutions.

Namely these solutions are written in the form

$$U(X)=e^{i\langle X_1\zeta_1+X_2\xi_2+\cdots+X_n\xi_n\rangle}, \tag{3}$$

where $\zeta_1\in C$ and $(\xi_2,\cdots,\xi_n)\in R^{n-1}\frown U(\xi_2^0,\cdots,\xi_n^0)$ for an arbitrary fixed neighbourhood $U(\xi_2^0,\cdots,\xi_n^0)$ of an arbitrary fixed point $(\xi_2^0,\cdots,\xi_n^0)\in R^{n-1}$.

We have already found the applications of this result in many different directions. Without proof we shall show the result which we have gotten, and we shall show the outline of the proof with an example. Afterwards we shall treat system's case in (II) and (III).

§ 2. Let's consider the function $U(X)=f(X)\,e^{i\langle X\zeta\rangle}$ where $\zeta\in C^n$ and $f(X)$ is a polynomial.

Lemma 1. If $U(X)$ is the solution of the equation $P(D)U=0$, then $P^{(\alpha)}(\zeta)=0$ for α by which $D_\alpha f(X)\neq 0$.

Let's consider the polynomial with complex coefficient, $P(\zeta)=a_0\zeta_1^k+a_1(\zeta_2,\cdots,\zeta_n)\zeta_1^{k-1}+\cdots+a_k(\zeta_1,\cdots,\zeta_n)$.