139. *Some Results in Lebesgue Geometry of Curves*

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1. **Borel-rectifiability of a curve on a set.** We shall resume the study of measure-theoretic properties of parametric curves set forth in our recent notes [4] and [5]. A curve φ, situated in a Euclidean space $\boldsymbol{R}^m$ of any dimension, will be said to be *Borel-rectifiable* (or *B-rectifiable*, for short) on a set E of real numbers, when and only when E admits an expression as the join of a sequence of sets which, if E is nonvoid, are relatively Borel with respect to E and on each of which φ is rectifiable. In other words, E can be covered by a sequence of Borel sets (in the absolute sense) on each of whose intersections with E the curve φ is rectifiable. As may be immediately seen, *this is certainly the case when φ is countably rectifiable on E and at the same time continuous on E.*

We are now in a position to generalize the theorem of [5]§3 to the following form, the proof being the same as before.

THEOREM. *For each function $f(t)$ which is Borel-rectifiable on a Borel set E, the multiplicity $N(f; x; E)$ is a measurable function of x and its integral over the real line coincides with $\Xi(f; E)$ and with $\Gamma(f; E)$.*

Moreover, an inspection of part 2) of the proof for the theorem of [5]§2 leads readily to the following extension of that theorem.

THEOREM. *If a curve φ is Borel-rectifiable on a set E, then $\Xi(\varphi; E)$ coincides with $\Gamma(\varphi; E)$.*

Let us make a few remarks. The function $f(t)$, defined to be 0 or 1 according as t is rational or irrational, gives an example to the last theorem when we consider the unit interval $I=[0,1]$ for instance. Since $f(t)$ is neither continuous on I nor rectifiable (i.e. of bounded variation) on I, this case is not covered by the theorem of [5]§2. On the other hand we cannot decide at present whether B-rectifiability may be replaced in our result by countable rectifiability or by a still weaker condition. But we can at least assert that B-rectifiability of φ on E is not always necessary for the coincidence of $\Xi(\varphi; E)$ and $\Gamma(\varphi; E)$.

In fact, put $I=[0,1]$ as above and choose a non-measurable set $A\subset I$. Then the characteristic function of the set A, for which we shall write $g(t)$, is obviously countably rectifiable (that is, VBG) on I and we find immediately that $\Xi(g; I)=\Gamma(g; I)=0$. We proceed to