18. On a Maximum Principle for Quasi-linear Elliptic Equations

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1. Introduction. In this note we shall consider second order quasi-linear elliptic equations of the form

$$\text{(A)} \qquad \sum_{i,j=1}^{n} a_{ij}(x, u, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u)^{1),2)}$$

whose solutions $u(x)$ are assumed to exist and to be of class C^2 in some domain G.

The purpose of this paper is to establish a maximum principle for solutions of the equations (A) under comparatively mild assumptions so as to extend the classical maximum principles.[3] Once the maximum principle has been established, our next task is to exhibit some of its applications. Thus, for instance, the uniqueness of the solution of the Dirichlet problem for some quasi-linear elliptic equations will be proved.

We can show, in view of the similarity lying between elliptic and parabolic equations, the validity of an analogous maximum principle for quasi-linear parabolic equations of the second order. However, its description will be left to another opportunity.

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2. Maximum principle. We shall begin with the simplest and the most evident fact concerning the maximum principle for the equation (A).

Proposition. *Let the following conditions be satisfied:*

i) *The quadratic form* $\sum_{i,j=1}^{n} a_{ij}(x, u, 0)\xi_i\xi_j$ *is positive definite for every x and u under consideration.*

ii) *The function $f(x, u, 0)$ is positive for positive u.*

Then any solution $u(x) \in C^2(G)$ of the equation (A) *cannot assume its positive maximum in the interior of G.*

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1) $x=(x_1,\cdots,x_n)$ and $\operatorname{grad} u=(\partial u/\partial x_1,\cdots,\partial u/\partial x_n)$.

2) The functions $a_{ij}(x,u,p)$ and $f(x,u,p)$ are defined in some domain in the space $(x,u,p)=(x_1,\cdots,x_n,u,p_1,\cdots,p_n)$.

3) See, e.g., Miranda [3], pp. 3–5.