25. On Theorems of Korovkin

By Hisashi CHODA and Marie ECHIGO
Department of Mathematics, Osaka Gakugei Daigaku

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1. In a recently published book [3], P. P. Korovkin established the following interesting theorems which are fundamental in his theory of approximation:

THEOREM 1. *If the two conditions*

(1) $$\sigma_n(1)\to 1, \quad \textit{as } n\to\infty,$$

(2) $$\sigma_n(g)\to 0, \quad \textit{as } n\to\infty,$$

where $a\leqq c\leqq b$ *and*

(3) $$g(x)=(x-c)^2,$$

are satisfied for the sequence of positive linear functionals σ_n *on the Banach space* $C[a, b]$ *of all continuous functions on* $[a, b]$, *then*

(4) $$\lim_{n\to\infty}\sigma_n(f)=f(c)$$

for any $f\in C[a, b]$.

THEOREM 2. *If the two conditions* (1) *and* (2) *are satisfied for the sequence of positive linear functionals* σ_n *on* $C[a, b]$ *and*

(5) $$g(x)=\sin^2\frac{x-c}{2},$$

where $a\leqq c\leqq b$, *then* (4) *is true for* $f\in C[a, b]$ *which has the period* 2π.

In this paper, we shall prove an abstract theorem which is a generalization of these theorems of Korovkin.

2. We shall introduce a few terms before we state our theorem. If a commutative Banach algebra A has an involution $x\to x^*$ satisfying $||xx^*||=||x||^2$ for any element x of A, then A will be called a *commutative B^*-algebra.* If a linear functional σ on a B^*-algebra A satisfies the condition that $\sigma(xx^*)\geqq 0$ for any element x of A, we shall say that σ is *positive.* It is well-known [4; p. 213] that a positive linear functional σ on a B^*-algebra A satisfies the inequality of Cauchy-Schwarz:

$$|\sigma(x^*y)|^2\leqq\sigma(|x|^2)\sigma(|y|^2)$$

for any $x, y\in A$, where $|x|=(x^*x)^{\frac{1}{2}}$. We shall call a positive linear functional σ a *state* whenever $\sigma(1)=1$ where 1 is the identity element of A. If a state χ of a commutative B^*-algebra is not expressible by a convex sum of two other states, χ will be called a *character.* It is also well-known [4; p. 229], that a character χ determines a maximal ideal M uniquely such as $M=\{x: \chi(x)=0\}$, and conversely that a maximal ideal M determines a character χ uniquely such that $\chi(x)$ coincides with the natural homomorphism of A onto A/M. Henceforth