37. On Completeness of Royden's Algebra

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Let R be a Riemann surface and $\boldsymbol{M}(R)$ be Royden's algebra associated with R, i.e. the totality of bounded continuous a.c.T. functions[*)] on R with finite Dirichlet integrals. We say that a sequence $\{\varphi_n\}$ of functions in $\boldsymbol{M}(R)$ converges to a function φ in C-topology if it converges uniformly on any compact subset of R. If a sequence $\{\varphi_n\}$ is bounded and converges to φ in C-topology, then we say that $\{\varphi_n\}$ converges to φ in B-topology. If the Dirichlet integral $\iint_R d(\varphi_n-\varphi)\wedge{}^*\overline{d(\varphi_n-\varphi)}$ tends to zero, then we say that $\{\varphi_n\}$ converges to φ in D-topology. Finally a sequence $\{\varphi_n\}$ converges to φ in BD-topology, if it converges in B-topology and D-topology. Let $\boldsymbol{M}_0(R)$ be the totality of functions in $\boldsymbol{M}(R)$ with compact supports in R and $\boldsymbol{M}_\Delta(R)$ be the potential subalgebra of $\boldsymbol{M}(R)$, i.e. the closure of $\boldsymbol{M}_0(R)$ in BD-topology. Let $\Gamma(R)$ be the totality of differentials α of the first order on R with finite Dirichlet integrals. Then $\Gamma(R)$ is a Hilbert space with an inner product $(\alpha,\beta)=\iint_R \alpha\wedge{}^*\bar{\beta}$. Clearly $\{df;\ f\in\boldsymbol{M}(R)\}\subset\Gamma(R)$. The algebras $\boldsymbol{M}(R)$ and $\boldsymbol{M}_\Delta(R)$ are complete with respect to BD-topology respectively. (cf. Lemma 1.5, p. 208 in Nakai [3]). Moreover we have the following theorem.

Theorem 1. *If $\varphi_n\in\boldsymbol{M}(R)$ and if* (1) *$\varphi_n\to\varphi$ in C-topology and φ is bounded,* (2) *the Dirichlet integral $D_R(\varphi_n)$ is bounded, then* (3) $\varphi\in\boldsymbol{M}(R)$, (4) $d\varphi_n\to d\varphi$ weakly in $\Gamma(R)$.

Proof. Generally, a bounded subset of a Hilbert space is weakly compact (cf. ch. 1, § 4 in Nagy [2]). Since $\{d\varphi_n\}$ is bounded in $\Gamma(R)$ by condition (2), there exists a subsequence $\{d\varphi_{n_k}\}$ such that $\{d\varphi_{n_k}\}$ converges to some $\alpha\in\Gamma(R)$ weakly in $\Gamma(R)$. We shall show that $\varphi\in\boldsymbol{M}(R)$ and $d\varphi=\alpha$. Let $z=x+iy$ be a local parameter in R and let G be a square domain: $-1<x<1$, $-1<y<1$ in the coordinate neighborhood of z. We put $\alpha=a(x,y)dx+b(x,y)dy$ in G and we take a differential β such that $\beta=\bar{\phi}dy$ in G and $\beta=0$ outside of G, where ϕ is in the class C^∞ and its support is contained in G. Then we have

$$(\alpha,\beta)=\iint \alpha\wedge{}^*\bar{\beta}=\iint_G a\phi\,dxdy.$$

By integration by parts, we get

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*) For the definition of a.c.T. functions, refer to A. Pfluger: Comment. Math. Helvt., **33**, 23-33 (1959).