35. *Representation of a Semigroup by Row-Monomial Matrices over a Group*

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Let G be a group written multiplicatively. An $n\times n$ matrix (where n can be any cardinal number) having at most one element of G in each row and zeros elsewhere is called a *row-monomial matrix* over G. The set $M(G, n)$ of all such matrices forms a semigroup under matrix multiplication. Schützenberger [2, 3] and Preston [1] have constructed *representations* of a semigroup S by row-monomial matrices, i.e., homomorphisms of S into $M(G, n)$. The purpose of this note is to present, without proofs, a new method for constructing such representations, which is more general than the methods used by Schützenberger and Preston.

Our method is similar to that used in the theory of monomial-representations of a group, and is somewhat analogous to the use, in ring theory, of modules over a ring R to construct representations of R by matrices over a field. We begin by defining the concept of a set with a semigroup S of operators (which, as in [4], we shall call an operand over S), and the endomorphisms of such sets (Section 1). In Section 2 we study a special class of operands, called free operands-with-zero, over a group G. These might be regarded as analogous to vector spaces. $M(G, n)$ is always isomorphic to the semigroup of endomorphisms of some free operand-with-zero over G. This leads in Section 3 to a procedure for determining all row-monomial representations of a semigroup. However, this result is not completely satisfactory, since it expresses the representations in terms of operands over S and their endomorphisms.

In Section 4, we restrict ourselves to a special kind row-monomial representation, viz., those in which at least one row can be "filled arbitrarily" [or "filled almost arbitrarily"]. This means that there is one row (say the i-th row) such that every monomial row vector [or every non-zero monomial row vector] actually occurs as the i-th row of one of the matrices corresponding to the elements of S. Thus the property in question is a kind of density condition.

It turns out that row monomial representations in which one row can be filled arbitrarily arise from strictly cyclic operands (in

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