34. A Characterization of Finite Projective Linear Groups

By Tosiro TSUZUKU

Mathematical Institute, Nagoya University

(Comm. by Kenjiro SHODA, M.J.A., March 12, 1964)

We say that a group G admits a Bruhat decomposition if it satisfies the following conditions (1) through (11).[1]

(1) G has three subgroups U, H, and W.

(2) H normalizes U.

(3) H is a normal subgroup of W.

(4) $W/H=\mathfrak{W}$ is a finite group.

(5) For each $w\in\mathfrak{W}$, U has subgroups U'_w and U''_w such that $U=U'_wU''_w$. For each $w\in\mathfrak{W}$, choose a representative $\omega(w)$ in W.

(6) $\omega(w)^{-1}U'_w\omega(w)\subset U$.

(7) $G=\sum_{w\in\mathfrak{W}} HU\omega(w)U''_w$, and in the representation $g=hu\omega(w)u''$ with $h\in H$, $u\in U$, $w\in\mathfrak{W}$ and $u''\in U''_w$, each factor is unique.

(8) There is a distinguished set $\mathfrak{F}$ of elements of $\mathfrak{W}$ of period 2 which generates $\mathfrak{W}$.

(9) For $w\in\mathfrak{F}$ and $s\in\mathfrak{W}$, $U''_w\subseteq U'_s$ implies $U''_w\subseteq U'_{sw}$.

(10) For $w\in\mathfrak{F}$, $HU+HU\omega(w)U''_w$ is a subgroup of G.

(11) There is an $x\in U$ such that $x\in U'_w$ implies $w=1$.

The group $\mathfrak{W}$ is called Weyl group associated to this Bruhat decomposition and the set $\mathfrak{F}$ is called the canonical set of generators of $\mathfrak{W}$.

It is well known that the projective special linear group $PSL(n,q)$, operating on the Desarguesian projective space of dimension $n-1$ over Galois field $GF(q)$, admits a Bruhat decomposition with the symmetric group S_n of degree n as Weyl group and with the set $\mathfrak{F}=\{(1,2),(2,3),\cdots,(n-1,n)\}$ as the canonical set of generators of $\mathfrak{W}$, where elements of S_n operate on n letters $1,2,\cdots,n$.

Recently D. G. Higman and J. E. Maclaughlin proved in [2] that a finite group G admitting a Bruhat decomposition with the symmetric group of degree 3 as Weyl group has a representation θ on a finite Desarquesian projective plane such that $\theta(G)$ contains the group $PSL(3,q)$. As a generalization of this theorem, we prove the following

Theorem. *Let a finite group G admit a Bruhat decomposition with the symmetric group S_n of degree $n\geqq 4$ as Weyl group $\mathfrak{W}$ and with the canonical set $\mathfrak{F}=\{(1,2),\cdots,(n-1,n)\}$ of generators of $\mathfrak{W}$ where elements of S_n operate on n letters $1,\cdots,n$. Then G has a representation θ on a finite projective space such that $\theta(G)$ contains*

1) See R. Steinberg [4].