# 88. On Subadditive Functionals and Linear Functionals on Abelian Group

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Recently D. Milman [2] has proved an interesting theorem relating with Hahn-Banach theorem. In this note, we shall consider his result on an Abelian group.

Let $G$ be an Abelian group, and consider a subadditive functional $p(x)$ on $G$, i.e. $p(x+y)\leq p(x)+p(y)$ and $p(0)=0$.

Define an order $p_2\prec p_1$ if $p_2(x)\leq p_1(x)$ for every $x\in G$. We have a well known theorem by Aumann: There is a linear functional $f(x)$, i.e. $f(x+y)=f(x)+f(y)$ such that $f\prec p$ for each subadditive $p$ (see K. Iséki [1]). Now we have the like of Milman result.

Theorem. *Let $p(x)$ be a subadditive functional, not linear functional. Then there is at least one minimal element for $p$ on the order $\prec$, and its element is linear. The set consisting of all elements of linear functionals $f$ such that $f\prec p$ coincides with the total set of minimal elements for $p$.*

To prove Theorem, we shall use a similar technique by D. Milman.

Proof. Since $p(x)$ is not linear, there are two elements $x_1, y_1$ such that $p(x_1+y_1)<p(x_1)+p(y_1)$. Let $H$ be the subgroup generated by $x_1, y_1$, then by Aumann theorem, there is a linear functional $f(x)$ on $H$ such that $f(x)\leq p(x)$ for $x\in H$. Put

$$p_1(x)=\inf_{y\in H}\{f(y)+p(x-y)\}$$

for $x\in G$. Then $-p(-x)\leq f(y)+p(x-y)$ implies $-p(-x)\leq p_1(x)\leq p(x)$. Therefore $p_1(x)$ is well-defined on $G$. Further, we have $p_1(x+y)\leq p_1(x)+p_1(y)$ and $-p(-y)\leq f(y)\leq p(y)$ for $y\in H$ implies $p_1(0)=0$. On the other hand, from

$$f(x_1)+f(y_1)=f(x_1+y_1)\leq p(x_1+y_1)<p(x_1)+p(y_1)$$

and $f(x_1)\leq p(x_1)$, $f(y_1)\leq p(y_1)$, for example, we have $p(x_1)-f(x_1)>0$. Hence

$$p_1(x_1)\leq f(x_1)+p(x_1-x_1)=f(x_1)<p(x_1),$$

and so $p_1\neq p$ and $p_1\prec p$.

If $\{p_\alpha\}$ is totally ordered set, then $p=\inf_\alpha p_\alpha$ is well-defined and subadditive on $G$. Hence at least one minimal element $p$ exists by Zorn's lemma. Suppose that $p$ is not linear, then there is a subadditive functional $p_1$ such that $p_1\prec p$ by the fiirst step of the proof. This is a contradiction.

If $f$ is linear and $p_1\prec f$, then we have $f(x)=-f(-x)\leq -p_1(-x)\leq$