86. A Note on Statistical Metric Spaces

By Itaru KUROSAKI
Osaka University of Liberal Arts and Education
(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1964)

1. Introduction. Statistical metric space is a space in which a probability distribution function $F_{pq}(x)$ is associated with each pair (p, q) of its points, while in a metric space a definite non-negative number is made to correspond to each pair. Some restrictions like the axioms of distances in metric spaces should be placed on the distribution functions of statistical metric spaces. One can find a typical formulation of these conditions together with a brief history of this kind of spaces and the references in [1] by B. Schweizer and A. Sklar. Several interesting results obtained by these authors and by E. Thorp are found also in [2], [3], [4], and [5]. In the series of these treatises, an axiom

$$F_{pr}(x+y) \geqq T(F_{pq}(x), F_{qr}(y)) \tag{1}$$

plays an important role throughout, where $T(u, v)$ is a function defined in the unit square and satisfies conditions such as

$$T(a, b)=T(b, a),\ T(a, 1)=a,\ T(0, 0)=0 \quad \text{etc.}$$

This axiom corresponds to the triangular inequality in the metric spaces. Now here we have an important problem how to define a topology in a statistical metric space S. In the papers listed above, a topological structure of S is given by the system of neighbourhoods $\{N_p(\varepsilon, \lambda)\}$, where

$$N_p(\varepsilon, \lambda)=\{q;\ F_{pq}(\varepsilon)>1-\lambda\}.$$

This scheme is closely connected to the convention

$$F_{pp}(x) \equiv H(x)=\begin{cases}1 & x>0 \\ 0 & x \leqq 0\end{cases}.$$

It is to be noted that S is a Hausdorff space with the system of neighbourhoods $\{N_p(\varepsilon, \lambda)\}$ and $F_{pq}(x)$ is continuous with respect to (p, q). But some additional assumptions other than the conditions on t-functions are necessary to obtain the latter result.

In this note, we first define a topology in the space of distribution functions and then reflecting this structure, a topology will be introduced into S to the effect that S is a Hausdorff space and F_{pq} is continuous with respect to (p, q) without t-functions.

2. Topology in the space of distribution functions. $\mathfrak{F}$ is the set of all functions $F(x)$ of a real variable x having properties i) non-negative valued, ii) monotone increasing, iii) right continuous, iv) $F(x)=0$ for $x<0$, and v) $F(x)\to 1$ as $x \to +\infty$.