82. On a Definition of Singular Integral Operators. II

By Hitoshi KUMANO-GO
Department of Mathematics, Osaka University
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2. Main theorems. In this part we shall prove that the main theorems relating to singular integral operators in the sense of A. P. Calderón and A. Zygmund [1] holds for ours defined in the part I by using the lemmas proved there.

Theorem 1. *$H\in S(\lambda, T_s)$ defined in Definition 4 in the part I is a bounded operator in L^2_x and*

$$\|Hu\|\leqq\{\delta'/(\delta'-\delta)\}^s\Big(\sum_{i=1}^k A_i\Big)\|u\|,\quad u\in L^2_x, \tag{2.1}$$

where $A_i=\sup_{R^n\times\mathcal{D}^*(\eta^{(i)},\delta')}|h_i(x,\zeta)|\cdot\sup_\eta|\alpha_i(\eta)|$.

Proof. In the representation (1.16) we have for $u\in L^2_x$

$$\|a_i^{(\nu)}H_i^{(\nu)}u\|\leqq\sup_{R^n\times\mathcal{D}(\eta^{(i)},\delta)}|a_i^{(\nu)}(x)(\eta-\eta^{(i)})^\nu|\cdot\sup_\eta|\alpha_i(\eta)|\cdot\|u\|.$$

Hence by (1.14) we have $\|a_i^{(\nu)}H_i^{(\nu)}u\|\leqq(\delta/\delta')^{|\nu|}A_i\|u\|$, and therefore

$$\|Hu\|\leqq\sum_{i=1}^k\sum_\nu(\delta/\delta')^{|\nu|}A_i\|u\|=\{\delta'/(\delta'-\delta)\}^s\Big(\sum_{i=1}^kA_i\Big)\|u\|.\quad \text{Q.E.D.}$$

Theorem 2. *Let $H\in S(\lambda, T_s)$ and $\Gamma\in T(p)$, $-\infty<p<+\infty$. Then, for any $\sigma_0\geqq0$ we have the representation*

$$\begin{aligned}\Gamma H-H\Gamma&=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}H_\alpha\cdot(x^\alpha\Gamma)+K^{(1)}_{\sigma_0}\\&=-\sum_{1\leqq|\alpha|\leqq l-1}\frac{1}{\alpha!}(x^\alpha\Gamma)\cdot H_\alpha+K^{(2)}_{\sigma_0}\end{aligned}\tag{2.2}$$

for every $l>\text{Max}\,[\{(4k+n)\tau+p\}/\rho,0]$ with $k=[\sigma_0/(2\rho)+1]$, where $H_\alpha\in S(\lambda, T_s)$ defined by $\sigma(H_\alpha)(x,\eta)=D^\alpha_x\sigma(H)(x,\eta)$ and $K^{(i)}_{\sigma_0}$ $(i=1,2)$ are of order σ_0 such that

$$\|\Lambda^{\sigma_1}K^{(i)}_{\sigma_0}\Lambda^{\sigma_2}\|$$
$$\leqq C_{\sigma_0,l,\gamma}\Big(\frac{\delta'}{\delta'-\delta}\Big)^s\sum_{i=1}^k\Big\{\sum_{|\beta|\leqq4k+l}\sup_{R^n\times\mathcal{D}^*(\eta^{(i)},\delta')}|D^\beta_xh_i(x,\zeta)|\cdot\sup_\eta|\alpha_i(\eta)|\Big\}.$$

Corollary. *If $H\in S(\lambda, T_s)$ and $\Psi\overset{\infty}{\equiv}0$, then $H\Psi\overset{\infty}{\equiv}\Psi H\overset{\infty}{\equiv}0$.*

Proof. By (1.16) and (1.17) we have $\Gamma H-H\Gamma=\sum_{i=1}^k\sum_\nu(\Gamma a_i^{(\nu)}-a_i^{(\nu)}\Gamma)H_i^{(\nu)}$ and by Lemma 3

$$\Gamma a_i^{(\nu)}-a_i^{(\nu)}\Gamma=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}D^\alpha_xa_i^{(\nu)}\cdot(x^\alpha\Gamma)+K^{(\nu)}_{\sigma_0,i}\equiv I_i^{(\nu)}+K^{(\nu)}_{\sigma_0,i}.$$

It is easy to see

$$\sum_{i=1}^k\sum_\nu I_i^{(\nu)}=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}H_\alpha\cdot(x^\alpha\Gamma).$$